

An Ergodic Study of Painlevé VI*

Katsunori Iwasaki and Takato Uehara

Graduate School of Mathematics, Kyushu University
6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8581 Japan[†]

Dedicated to Professor Masuo Hukuhara on his 100th birthday

Abstract

An ergodic study of Painlevé VI is developed. The chaotic nature of its Poincaré return map is established for almost all loops. The exponential growth of the numbers of periodic solutions is also shown. Principal ingredients of the arguments are a moduli-theoretical formulation of Painlevé VI, a Riemann-Hilbert correspondence, the dynamical system of a birational map on a cubic surface, and the Lefschetz fixed point formula.

1 Introduction

Painlevé equations have been investigated actively in recent years. However most researches have been done from the viewpoint of integrable systems and little attention has been paid to the ergodic and chaotic aspects of their dynamics. In this paper we develop an ergodic study of the sixth Painlevé equation $P_{VI}(\kappa)$ and explore the chaotic behavior of its global dynamics, namely, that of its Poincaré return map. The aim of this paper is to show that the Poincaré return map is chaotic along almost all loops in the space of independent variable

$$Z = \mathbb{P}^1 - \{0, 1, \infty\}.$$

The exponential growth of the number of periodic solutions along those loops is also established.

The sixth Painlevé equation $P_{VI}(\kappa)$ is a Hamiltonian system of differential equations

$$\frac{dq}{dz} = \frac{\partial H(\kappa)}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial H(\kappa)}{\partial q}, \quad (1)$$

with an independent variable $z \in Z$ and unknown functions $q = q(z)$ and $p = p(z)$, depending on complex parameters $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ in a 4-dimensional affine space

$$\mathcal{K} := \{ \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 : 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \}, \quad (2)$$

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[†]E-mail addresses: iwasaki@math.kyushu-u.ac.jp and ma205003@math.kyushu-u.ac.jp

where the Hamiltonian $H(\kappa) = H(q, p, z; \kappa)$ is given by

$$z(z-1)H(\kappa) = (q_0q_1q_z)p^2 - \{\kappa_1q_1q_z + (\kappa_2-1)q_0q_1 + \kappa_3q_0q_z\}p + \kappa_0(\kappa_0 + \kappa_4)q_z,$$

with $q_\nu = q - \nu$ for $\nu \in \{0, 1, z\}$. More intrinsically, $P_{VI}(\kappa)$ can be formulated as a holomorphic uniform foliation on a fibration of certain smooth quasi-projective rational surfaces

$$\pi_\kappa : M(\kappa) \rightarrow Z, \tag{3}$$

which is transversal to each fiber of the fibration. Equation (1) is just a coordinate expression of the foliation in terms of a natural coordinate system on an affine open subset of the phase space $M(\kappa)$. See [1, 16, 17, 28, 30, 31] for various construction of the space $M(\kappa)$. Especially the papers [16, 17] give a comprehensive description of it as a moduli space of stable parabolic connections. The fiber $M_z(\kappa)$ over $z \in Z$ is called the space of initial conditions at time z .

Since the foliation is uniform (Painlevé property), each loop $\gamma \in \pi_1(Z, z)$ admits global horizontal lifts along the foliation and induces an automorphism $\gamma_* : M_z(\kappa) \rightarrow M_z(\kappa)$, called the holonomy or the Poincaré return map along the loop γ . Then the global structure of the foliation is described by the holonomy representation

$$PS_z(\kappa) : \pi_1(Z, z) \rightarrow \text{Aut } M_z(\kappa), \quad \gamma \mapsto \gamma_*. \tag{4}$$

which is referred to as the Poincaré section of the Painlevé dynamical system $P_{VI}(\kappa)$. Here and hereafter a *loop* means the homotopy class of a loop without further comment.

In this paper we are interested in the dynamics of the Poincaré return map $\gamma_* : M_z(\kappa) \rightarrow M_z(\kappa)$ for each individual loop $\gamma \in \pi_1(Z, z)$. One of our main results will state that γ_* always exhibits a chaotic behavior as long as γ is a non-elementary loop (see Theorem 2.1), where the adjective “chaotic” and the words “non-elementary loop” are used in the following senses.

Definition 1.1 The dynamical system of a holomorphic map $f : S \rightarrow S$ on a complex surface S (in our case, $S = M_z(\kappa)$ and $f = \gamma_*$) is said to be *chaotic* if there exists an f -invariant Borel probability measure μ on S such that the following conditions are satisfied:

- (C1) f has a positive entropy $h_\mu(f) > 0$ with respect to the measure μ .
- (C2) f is mixing with respect to the measure μ , that is, $\mu(f^{-n}(A) \cap B) \rightarrow \mu(A)\mu(B)$ as $n \rightarrow \infty$ for any Borel subsets A and B of S . In particular, f is ergodic with respect to μ .
- (C3) μ is a hyperbolic measure of saddle type, that is, the two Lyapunov exponents $L_\pm(f)$ of f with respect to the ergodic measure μ satisfy $L_-(f) < 0 < L_+(f)$. Moreover, μ has product structure with respect to local stable and unstable manifolds.
- (C4) hyperbolic periodic points of f are dense in the support of μ .

For the basic terminology used here, we refer to the standard textbooks [24, 32] on dynamical systems and ergodic theory. While there are many possible definitions of a chaotic dynamical system [6], the definition adopted here is a typical one possessing the three ingredients usually required for a “chaos”: (i) unpredictability, that is, the sensitive dependence on initial values represented by conditions (C1) and (C3); (ii) indecomposability, that is, ergodicity or a related property as in (C2); (iii) an element of regularity, that is, the existence of periodic points which

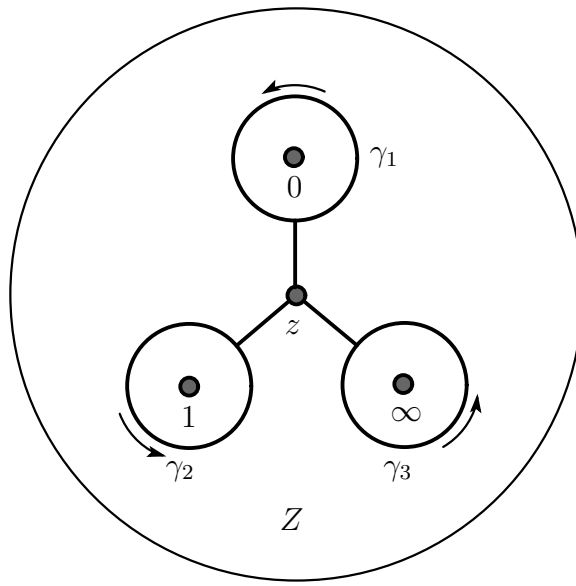


Figure 1: Three basic loops in $Z = \mathbb{P}^1 - \{0, 1, \infty\}$

are dense in a dynamically interesting subset as in (C4). We also remark that conditions (C2) and (C3) imply that the dynamical system f with invariant measure μ is Bernoulli, namely, it is measurably conjugate to a Bernoulli shift [29].

Next we explain what we mean by the words “non-elementary loop”. Treating the three fixed singular points $0, 1, \infty$ of $P_{VI}(\kappa)$ symmetrically, we put

$$z_1 = 0, \quad z_2 = 1, \quad z_3 = \infty.$$

For each $i \in \{1, 2, 3\}$ with $\{i, j, k\} = \{1, 2, 3\}$, let $\gamma_i \in \pi_1(Z, z)$ be a loop surrounding the points z_i once anti-clockwise, leaving the remaining points z_j and z_k outside, as in Figure 1. Then the fundamental group $\pi_1(Z, z)$ is generated by $\gamma_1, \gamma_2, \gamma_3$, having a defining relation

$$\gamma_1 \gamma_2 \gamma_3 = 1. \quad (5)$$

Definition 1.2 A loop $\gamma \in \pi_1(Z, z)$ is said to be *elementary* if γ is conjugate to the loop γ_i^m for some $i \in \{1, 2, 3\}$ and $m \in \mathbb{Z}$. Otherwise, γ is said to be *non-elementary*.

The second issue to be discussed in this paper is the number of periodic solutions to $P_{VI}(\kappa)$. Given a loop $\gamma \in \pi_1(Z, z)$ and a positive integer $N \in \mathbb{N}$, we are interested in the number of periodic solutions to $P_{VI}(\kappa)$ of period N along the loop γ . To be more precise, we wish to count the number of all initial conditions at time z that come back to the original positions after the N -th iterate of the Poincaré return map along γ , namely, the cardinality of the set

$$\text{Per}_N(\gamma; \kappa) = \{ Q \in M_z(\kappa) : \gamma_*^N Q = Q \}. \quad (6)$$

It will be shown that for any non-elementary loop γ , the cardinality is finite for every period $N \in \mathbb{N}$ and grows exponentially as the period N tends to infinity (see Theorem 2.2). We shall

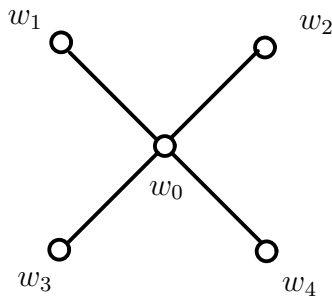


Figure 2: Dynkin diagram of type $D_4^{(1)}$

also give an algorithm to count the number exactly as well as to determine its exponential growth rate explicitly (see Theorem 2.8). The logarithm of this rate will give the entropy of the Poincaré return map γ_* . Recently several authors [4, 5, 9, 13, 14, 25, 26] have been interested in finding algebraic solutions, which must have only finitely many branches under the analytic continuations along *all* loops in Z . On the other hand, in this article we will be concerned with those solutions which are finitely many-valued along a fixed *single* loop.

Painlevé equations and dynamical systems on complex manifolds are two subjects of mathematics which have attracted much attention in recent years. In this paper we shall demonstrate a substantial interplay between them by presenting a fruitful application to the former subject of the latter. On the former side, algebraic geometry of Painlevé equations, especially a moduli-theoretical formulation of Painlevé dynamical systems [16, 17] is an essential ingredient of our discussion. On the latter side, recent advances in complex surface dynamics, especially some deep ergodic studies of birational maps of complex surfaces [2, 7, 8, 10] are another basis of our analysis. These two stuffs are combined fruitfully via a Riemann-Hilbert correspondence to reveal the chaotic nature of the sixth Painlevé dynamics.

2 Main Results

Let us describe our main results in more detail. In this paper we make a certain generic assumption on the parameters $\kappa \in \mathcal{K}$ to avoid a technical difficulty (see Remark 2.11). To this end we recall an affine Weyl group structure of the parameter space \mathcal{K} [16, 20]. In view of formula (2), the affine space \mathcal{K} can be identified with the linear space \mathbb{C}^4 by the isomorphism

$$\mathcal{K} \rightarrow \mathbb{C}^4, \quad \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \mapsto (\kappa_1, \kappa_2, \kappa_3, \kappa_4),$$

where the latter space \mathbb{C}^4 is equipped with the standard (complex) Euclidean inner product. For each $i \in \{0, 1, 2, 3, 4\}$, let $w_i : \mathcal{K} \rightarrow \mathcal{K}$ be the orthogonal reflection having $\{\kappa \in \mathcal{K} : \kappa_i = 0\}$ as its reflecting hyperplane with respect to the inner product mentioned above. Then the group generated by w_0, w_1, w_2, w_3, w_4 is an affine Weyl group of type $D_4^{(1)}$,

$$W(D_4^{(1)}) = \langle w_0, w_1, w_2, w_3, w_4 \rangle \curvearrowright \mathcal{K}.$$

corresponding to the Dynkin diagram in Figure 2. The reflecting hyperplanes of all reflections in the group $W(D_4^{(1)})$ are given by affine linear relations

$$\kappa_i = m, \quad \kappa_1 \pm \kappa_2 \pm \kappa_3 \pm \kappa_4 = 2m + 1 \quad (i \in \{1, 2, 3, 4\}, m \in \mathbb{Z}),$$

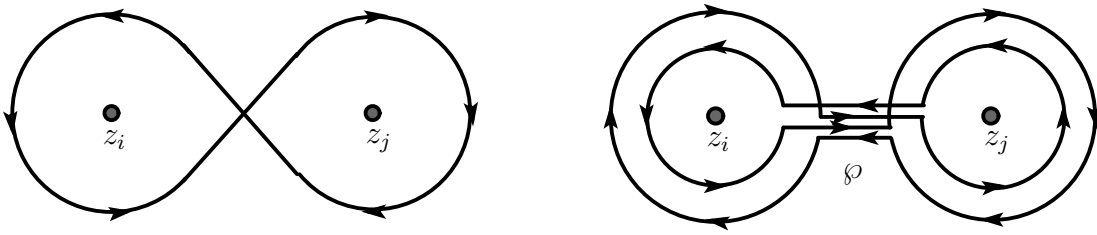


Figure 3: An eight-loop (left) and a Pochhammer loop (right)

where the signs \pm may be chosen arbitrarily. Let \mathbf{Wall} be the union of all these hyperplanes. Then the generic condition to be imposed on parameters is that κ should lie outside \mathbf{Wall} ; this is a necessary and sufficient condition for $P_{VI}(\kappa)$ to admit no Riccati solutions [16].

The first main theorem of this paper is concerned with the chaotic behavior of $P_{VI}(\kappa)$.

Theorem 2.1 *Assume that $\kappa \in \mathcal{K} - \mathbf{Wall}$. For any non-elementary loop $\gamma \in \pi_1(Z, z)$, the Poincaré return map $\gamma_* : M_z(\kappa) \curvearrowright$ along the loop γ is chaotic, that is, there exists a γ_* -invariant Borel probability measure μ_γ such that the conditions of Definition 1.1 are satisfied. Moreover there exists an algorithm to calculate the entropy $h(\gamma) := h_{\mu_\gamma}(\gamma_*)$ of the map γ_* with respect to the measure μ_γ in terms of a reduced word for the loop γ (see Theorem 2.8).*

The second main theorem is about the periodic solutions to $P_{VI}(\kappa)$ along a given loop.

Theorem 2.2 *Assume that $\kappa \in \mathcal{K} - \mathbf{Wall}$. For any non-elementary loop $\gamma \in \pi_1(Z, z)$, the cardinality of the set $\text{Per}_N(\gamma; \kappa)$ is finite for every period $N \in \mathbb{N}$ and grows exponentially as N tends to infinity. There is an algorithm to count the cardinality exactly as well as to determine its exponential growth rate in terms of a reduced word for the loop γ (see Theorem 2.8).*

Example 2.3 We illustrate Theorems 2.1 and 2.2 by presenting two examples.

- (1) An *eight-loop* is a loop conjugate to $\gamma_i \gamma_j^{-1}$ for some indices $\{i, j, k\} = \{1, 2, 3\}$ as in Figure 3 (left). For an eight-loop γ we have

$$h(\gamma) = \log(3 + 2\sqrt{2}), \quad \# \text{Per}_N(\gamma; \kappa) = (3 + 2\sqrt{2})^N + (3 + 2\sqrt{2})^{-N} + 4.$$

- (2) A *Pochhammer loop* is a loop conjugate to the commutator $[\gamma_i, \gamma_j^{-1}] = \gamma_i \gamma_j^{-1} \gamma_i^{-1} \gamma_j$ for some indices $\{i, j, k\} = \{1, 2, 3\}$ as in Figure 3 (right). For a Pochhammer loop \wp we have

$$h(\wp) = \log(9 + 4\sqrt{5}), \quad \# \text{Per}_N(\wp; \kappa) = (9 + 4\sqrt{5})^N + (9 + 4\sqrt{5})^{-N} + 4.$$

As is mentioned in Theorems 2.1 and 2.2, there are algorithms to calculate the entropy and to count the number of periodic solutions exactly. In order to describe them we need some preparations concerning reduced words for representing loops in terms of the standard generators $\gamma_1, \gamma_2, \gamma_3$.

Definition 2.4 For any nontrivial loop $\gamma \in \pi_1(Z, z)$, there exists an expression

$$\gamma = \gamma_{i_1}^{\varepsilon_{i_1}} \gamma_{i_2}^{\varepsilon_{i_2}} \cdots \gamma_{i_m}^{\varepsilon_{i_m}}, \quad (7)$$

with some positive number $m \in \mathbb{N}$, some indices $(i_1, \dots, i_m) \in \{1, 2, 3\}^m$ and some signs $(\varepsilon_{i_1}, \dots, \varepsilon_{i_m}) \in \{\pm 1\}^m$. Such an expression is not unique and its length m may be reduced by using the relation (5). The expression (7) is said to be *reduced* if its length m is minimal among all feasible expressions. The *length* $\ell_{\pi_1}(\gamma)$ of the loop γ is defined to be the length m of a reduced expression (7) for γ . By convention the length of the trivial loop is zero.

Remark 2.5 At this stage we should notice that relevant to our discussion is not a loop itself but the conjugacy class of a loop. Indeed, if two loops γ and γ' are conjugate to each other, say, $\gamma' = \delta\gamma\delta^{-1}$ for some loop δ , then the corresponding Poincaré return maps are also conjugate to each other as $\gamma'_* = \delta_*\gamma_*\delta_*^{-1}$, and hence have the same dynamical properties. If μ_γ is a γ_* -invariant measure asserted in Theorem 2.1, then the push-forward $\mu_{\gamma'} = (\delta_*)_*\mu_\gamma$ of the measure μ_γ by the map δ_* is a desired invariant measure for γ'_* . As for the sets of periodic points, the loop δ induces a bijection $\delta_* : \text{Per}_N(\gamma; \kappa) \rightarrow \text{Per}_N(\gamma'; \kappa)$ and hence an equality $\#\text{Per}_N(\gamma; \kappa) = \#\text{Per}_N(\gamma'; \kappa)$. So what is relevant is only the conjugacy class of a loop.

This remark leads us to the following definition.

Definition 2.6 A loop $\gamma \in \pi_1(Z, z)$ is said to be *minimal* if it has the minimal length among all loops conjugate to γ , namely, if $\ell_{\pi_1}(\gamma) = \min\{\ell_{\pi_1}(\gamma') : \gamma' \in \pi_1(Z, z) \text{ is conjugate to } \gamma\}$.

In what follows we may and shall consider minimal loops only by replacing a given loop with a minimal representative for the conjugacy class of the loop, if it is not a minimal loop.

In order to give the algorithm, we shall identify $\pi_1(Z, z)$ with an index-two subgroup of the universal Coxeter group of rank three, that is, the free product of three copies of $\mathbb{Z}/2\mathbb{Z}$,

$$G = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \rangle \simeq (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}).$$

Any element $\sigma \in G$ other than the unit element is uniquely represented in the form

$$\sigma = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}, \tag{8}$$

for some $n \in \mathbb{N}$ and some n -tuple of indices $(i_1, \dots, i_n) \in \{1, 2, 3\}^n$ such that every neighboring indices i_ν and $i_{\nu+1}$ are distinct. The expression (8) is called the *reduced* expression of σ and the number $\ell_G(\sigma) = n$ is called the *length* of σ , where the unit element is of length zero by convention. An element of even length is called an *even* element. Let $G(2)$ be the subgroup of all even elements in G . Then there exists an isomorphism of groups

$$\pi_1(Z, z) \rightarrow G(2) \tag{9}$$

sending the basic loops and their inverses as

$$\begin{aligned} \gamma_1 &\mapsto \sigma_1\sigma_2, & \gamma_2 &\mapsto \sigma_2\sigma_3, & \gamma_3 &\mapsto \sigma_3\sigma_1, \\ \gamma_1^{-1} &\mapsto \sigma_2\sigma_1, & \gamma_2^{-1} &\mapsto \sigma_3\sigma_2, & \gamma_3^{-1} &\mapsto \sigma_1\sigma_3. \end{aligned} \tag{10}$$

Given an expression (7) of a loop $\gamma \in \pi_1(Z, z)$, make the replacement of alphabets

$$\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \gamma_3^{\pm 1}\} \rightarrow \{\sigma_1, \sigma_2, \sigma_3\}$$

according to the rule (10). If the expression (7) is reduced in $\pi_1(Z, z)$, then the resulting word is also reduced in G . In particular the reduced expression (7) is unique for a given loop γ and one has $\ell_G(\sigma) = 2\ell_{\pi_1}(\gamma)$, where $\sigma \in G(2)$ is the element corresponding to the loop γ .

Recall that any Coxeter group admits its geometric representation [15]. We apply this construction to our particular group G . Let V be the 3-dimensional vector space spanned by basis vectors e_1, e_2, e_3 , endowed with a nondegenerate symmetric bilinear form

$$B(e_i, e_j) = \begin{cases} 1 & (i = j), \\ -1 & (i \neq j). \end{cases} \quad (11)$$

For each $i \in \{1, 2, 3\}$ we can define an orthogonal reflection $r_i : V \rightarrow V$ by the rule

$$r_i(v) := v - 2B(e_i, v) e_i \quad (v \in V). \quad (12)$$

Note that r_i sends e_i to its negative $-e_i$ while fixing all the vectors orthogonal to e_i relative to the bilinear form B . It is known that there is a unique injective homomorphism $\text{GR} : G \rightarrow \text{O}_B(V)$ such that $\text{GR}(\sigma_i) = r_i$ for $i \in \{1, 2, 3\}$, where $\text{O}_B(V)$ is the group of orthogonal transformations on (V, B) . Identified with its image $\text{GR}(G)$, the group G can be thought of as a reflection group acting on (V, B) . The faithful representation $\text{GR} : G \rightarrow \text{O}_B(V)$ is called the geometric representation of G . For each $i \in \{1, 2, 3\}$ we define an endomorphism $s_i : V \rightarrow V$ by

$$s_i(v) := \frac{v + r_i(v)}{2} = v - B(e_i, v) e_i \quad (v \in V), \quad (13)$$

and make the following definition.

Definition 2.7 Given a loop $\gamma \in \pi_1(Z, z)$, choose a minimal representative for the conjugacy class of γ and call it γ again. Take the reduced expression of γ as in (7). Make the change of alphabets $\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \gamma_3^{\pm 1}\} \rightarrow \{\sigma_1, \sigma_2, \sigma_3\}$ according to the rule (10) to obtain the corresponding element $\sigma \in G(2)$, together with its reduced expression as in (8). To the indices (i_1, \dots, i_n) in (8), associate an endomorphism $s_\gamma := s_{i_n} \cdots s_{i_2} s_{i_1} \in \text{End } V$. Finally, take its trace

$$\alpha(\gamma) = \text{Tr}[s_\gamma : V \rightarrow V]. \quad (14)$$

We are now in a position to give the algorithm to calculate the entropy and to count the number of periodic points, which complete the statements of Theorems 2.1 and 2.2.

Theorem 2.8 Assume that $\kappa \in \mathcal{K} - \mathbf{Wall}$ and let $\gamma \in \pi_1(Z, z)$ be any non-elementary loop. Then the number $\alpha(\gamma)$ defined in (14) is an even integer not smaller than 6, with the equality $\alpha(\gamma) = 6$ if and only if γ is an eight-loop as in Example 2.3. Put

$$\lambda(\gamma) := \frac{1}{2} \left\{ \alpha(\gamma) + \sqrt{\alpha(\gamma)^2 - 4} \right\}. \quad (15)$$

- (1) The measure-theoretic entropy $h(\gamma) := h_{\mu_\gamma}(\gamma_*)$ of the Poincaré return map $\gamma_* : M_z(\kappa) \curvearrowright$ with respect to the invariant measure μ_γ mentioned in Theorem 2.1 is given by

$$h(\gamma) = \log \lambda(\gamma).$$

- (2) The cardinality of the set $\text{Per}_N(\gamma; \kappa)$ is given by

$$\#\text{Per}_N(\gamma; \kappa) = \lambda(\gamma)^N + \lambda(\gamma)^{-N} + 4 \quad (N \in \mathbb{N}).$$

In particular its exponential growth rate is given by $\lambda(\gamma)$.

Remark 2.9 Theorem 2.8 implies that for any non-elementary loop $\gamma \in \pi_1(Z, z)$, we have

$$\lambda(\gamma) \geq 3 + 2\sqrt{2}, \quad h(\gamma) \geq \log(3 + 2\sqrt{2}),$$

with the equalities if and only if γ is an eight-loop. In this sense the eight-loops are the most “elementary” loops among all non-elementary loops in Z . On the other hand, one may ask what happens with the Poincaré return map $\gamma_* : M_z(\kappa) \curvearrowright$ when the loop γ is elementary. In this case it turns out that γ_* preserves a certain analytic fibration $M_z(\kappa) \rightarrow \mathbb{C}$ and exhibits an essentially 1-dimensional dynamical behavior. Hence γ_* is not so interesting or too elementary from the standpoint of chaotic dynamical systems. See Remark 10.3 for more information.

Remark 2.10 There exists a standard complex area form $\omega_z(\kappa)$ on $M_z(\kappa)$ such that the Poincaré return map γ_* is area-preserving for every loop $\gamma \in \pi_1(Z, z)$, where we refer to Remark 3.4 for the description of $\omega_z(\kappa)$. Hence the Lyapunov exponents $L_{\pm}(\gamma)$ of γ_* satisfy the relation $L_{-}(\gamma) = -L_{+}(\gamma)$. Moreover the positive Lyapunov exponent admits an estimate $L_{+}(\gamma) \geq \frac{1}{8} \log \lambda(\gamma)$. We refer to Remark 10.3 for the derivation of this estimate.

Remark 2.11 In this article we restrict our attention to the generic case $\kappa \in \mathcal{K} - \mathbf{Wall}$ only, leaving the nongeneric case $\kappa \in \mathbf{Wall}$ untouched. The difference between the generic case and the nongeneric case lies in the fact that the Riemann-Hilbert correspondence to be used in the proof becomes a biholomorphism in the former case, while it gives an analytic minimal resolution of Klein singularities in the latter case (see Remark 4.2). The presence of singularities would make the treatment of the nongeneric case more complicated. However it is expected that the basic strategy developed in this article will be effective also in the nongeneric case. The relevant discussion will be made elsewhere.

The plan of this article is as follows: $P_{VI}(\kappa)$ is formulated as a flow, *Painlevé flow*, on a moduli space of stable parabolic connections in §3. It is conjugated to an isomonodromic flow on a moduli space of monodromy representations via a Riemann-Hilbert correspondence in §4. The moduli space of monodromy representations is identified with an affine cubic surface and each Poincaré return map for $P_{VI}(\kappa)$ is conjugated to a biregular automorphism of the affine cubic in §5. This map is extended to a birational map on the compactified projective cubic surface and some basic properties of it are studied in §6. The induced cohomological action of the birational map is investigated in §7. After these preliminaries, the ergodic properties of our dynamical system are established by applying some recent deep results from birational surface dynamics in §8. Moreover the number of periodic points of the birational map is counted by using the Lefschetz fixed point formula in §9. Then, back to the original phase space of $P_{VI}(\kappa)$ in §10, we arrive at our final goals, that is, at the ergodic properties of the Poincaré return map and the exact number of periodic solutions to $P_{VI}(\kappa)$ of any period along a given loop.

3 Moduli Space of Stable Parabolic Connections

In order to describe the fibration (3), we first construct an auxiliary fibration $\pi_{\kappa} : \mathcal{M}(\kappa) \rightarrow T$ over the configuration space of mutually distinct, ordered, three points in \mathbb{C} ,

$$T = \{ t = (t_1, t_2, t_3) \in \mathbb{C}^3 : t_i \neq t_j \text{ for } i \neq j \},$$

and then reduce it to the original fibration (3). We put the fourth point t_4 at infinity. Given any $(t, \kappa) \in T \times \mathcal{K}$, a (t, κ) -parabolic connection is a quadruple $Q = (E, \nabla, \psi, l)$ such that

singularities	t_1	t_2	t_3	t_4
first exponent	$-\lambda_1$	$-\lambda_2$	$-\lambda_3$	$-\lambda_4$
second exponent	λ_1	λ_2	λ_3	$\lambda_4 - 1$
difference	κ_1	κ_2	κ_3	κ_4

Table 1: Riemann scheme: κ_i is the difference of the second exponent from the first.

- (1) E is a rank 2 algebraic vector bundle of degree -1 over \mathbb{P}^1 ,
- (2) $\nabla : E \rightarrow E \otimes \Omega_{\mathbb{P}^1}^1(D_t)$ is a Fuchsian connection with pole divisor $D_t = t_1 + t_2 + t_3 + t_4$ and Riemann scheme as in Table 1, where $t_4 = \infty$ as mentioned above,
- (3) $\psi : \det E \rightarrow \mathcal{O}_{\mathbb{P}^1}(-t_4)$ is a horizontal isomorphism called a determinantal structure, where $\mathcal{O}_{\mathbb{P}^1}(-t_4)$ is equipped with the connection induced from $d : \mathcal{O}_{\mathbb{P}^1} \rightarrow \Omega_{\mathbb{P}^1}^1$,
- (4) $l = (l_1, l_2, l_3, l_4)$ is a parabolic structure, namely, l_i is an eigenline of $\text{Res}_{t_i}(\nabla) \in \text{End}(E_{t_i})$ corresponding to eigenvalue λ_i (whose minus is the first exponent $-\lambda_i$ in Table 1).

There exists a concept of stability for parabolic connections, with which the geometric invariant theory [27] can be worked out to establish the following theorem [16, 17].

Theorem 3.1 *For any $(t, \kappa) \in T \times \mathcal{K}$ there exists a fine moduli scheme $\mathcal{M}_t(\kappa)$ of stable (t, κ) -parabolic connections. The moduli space $\mathcal{M}_t(\kappa)$ is a smooth, irreducible, quasi-projective surface. As a relative setting over T , for any $\kappa \in \mathcal{K}$, there exists a family of moduli spaces*

$$\pi_\kappa : \mathcal{M}(\kappa) \rightarrow T \quad (16)$$

such that the projection π_κ is a smooth morphism having fiber $\mathcal{M}_t(\kappa)$ over $t \in T$.

In [16, 17] the moduli space $\mathcal{M}_t(\kappa)$ is compactified into a moduli space of stable parabolic phi-connections. Given any $(t, \kappa) \in T \times \mathcal{K}$, a *parabolic phi-connection* is roughly speaking a sextuple of data $Q = (E_1, E_2, \phi, \nabla, \psi, l)$ consisting of

- (1) a variant of connection $\nabla : E_1 \rightarrow E_2 \otimes \Omega_{\mathbb{P}^1}^1(D_t)$ over rank 2, degree -1 bundles on \mathbb{P}^1 ,
- (2) an $\mathcal{O}_{\mathbb{P}^1}$ -homomorphism $\phi : E_1 \rightarrow E_2$ (called a *phi-operator*), which may be *degenerate* or non-isomorphic, satisfying a generalized Leibniz rule

$$\nabla(fs) = \phi(s) \otimes df + f\nabla(s), \quad (s \in E_1, f \in \mathcal{O}_{\mathbb{P}^1}),$$

- (3) extra data of a determinantal structure ψ and a parabolic structure l .

We refer to [16, 17] for the complete definition. Very roughly the idea of compactification is as follows: If a parabolic connection is regarded as a “matrix-valued Schrödinger operator”, then a parabolic phi-connection may be thought of as a matrix-valued Schrödinger operator with a “matrix-valued Planck constant” ϕ which may be degenerate, namely, may be semi-classical. Then the moduli space $\mathcal{M}_t(\kappa)$ can be compactified by adding some semi-classical objects, that is, some parabolic phi-connections with degenerate phi-operator ϕ .

There exists a concept of *stability* for parabolic phi-connections, with which geometric invariant theory can be worked out to establish the following theorem [16, 17].

Theorem 3.2 *For any $(t, \kappa) \in T \times \mathcal{K}$ there exists a coarse moduli scheme $\overline{\mathcal{M}}_t(\kappa)$ of stable parabolic phi-connections. The moduli space $\overline{\mathcal{M}}_t(\kappa)$ is a smooth, irreducible, projective surface, having a unique effective anti-canonical divisor $\mathcal{Y}_t(\kappa)$. Under the natural embedding*

$$\mathcal{M}_t(\kappa) \hookrightarrow \overline{\mathcal{M}}_t(\kappa), \quad (E, \nabla, \psi, l) \mapsto (E, E, \text{id}, \nabla, \psi, l),$$

the space $\mathcal{M}_t(\kappa)$ is exactly the locus of $\overline{\mathcal{M}}_t(\kappa)$ where the phi-operator ϕ is isomorphic, and so

$$\mathcal{M}_t(\kappa) = \overline{\mathcal{M}}_t(\kappa) - \mathcal{Y}_t(\kappa).$$

The divisor $\mathcal{Y}_t(\kappa)$ on $\overline{\mathcal{M}}_t(\kappa)$ is called the vertical leaves at time t . There is the following realization of our moduli spaces [16, 17] (see Figure 4).

Theorem 3.3 *The compactified moduli space $\overline{\mathcal{M}}_t(\kappa)$ is isomorphic to an 8-point blow-up of the Hirzebruch surface $\Sigma_2 \rightarrow \mathbb{P}^1$ of degree 2, blown up at certain two points on each fiber over the points $t_1, t_2, t_3, t_4 \in \mathbb{P}^1$. The unique effective anti-canonical divisor on $\overline{\mathcal{M}}_t(\kappa)$ is given by*

$$\mathcal{Y}_t(\kappa) = 2E_0 + E_1 + E_2 + E_3 + E_4,$$

where E_0 is the strict transform of the section at infinity of the fibration $\Sigma_2 \rightarrow \mathbb{P}^1$, while E_1, E_2, E_3, E_4 are the strict transforms of the fibers over t_1, t_2, t_3, t_4 , respectively.

Remark 3.4 There is a meromorphic 2-form $\omega_t(\kappa)$ on $\overline{\mathcal{M}}_t(\kappa)$, holomorphic and nondegenerate on $\mathcal{M}_t(\kappa)$, whose pole divisor is given by the vertical leaves $\mathcal{Y}_t(\kappa)$ [16, 17, 30, 31]. It is unique up to constant multiples. This complex area form is just what we have mentioned in Remark 2.10. A further description of the area form $\omega_t(\kappa)$ will be given in Remark 5.1.

Now the fibration (3) is defined to be the pull-back of the fibration (16) by an injection

$$\iota : Z \hookrightarrow T, \quad z \mapsto (0, z, 1),$$

The group $\text{Aff}(\mathbb{C})$ of affine linear transformations on \mathbb{C} acts diagonally on the configuration space T and the quotient space $T/\text{Aff}(\mathbb{C})$ is isomorphic to Z , with the quotient map given by

$$r : T \rightarrow Z, \quad t = (t_1, t_2, t_3) \mapsto z = \frac{t_2 - t_1}{t_3 - t_1}. \quad (17)$$

The map r yields a trivial $\text{Aff}(\mathbb{C})$ -bundle structure of T over Z and the fibration (16) is in turn the pull-back of the fibration (3) by the map r . Hence we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\kappa) & \longrightarrow & M(\kappa) \\ \pi_\kappa \downarrow & & \downarrow \pi_\kappa \\ T & \xrightarrow{\quad r \quad} & Z. \end{array} \quad (18)$$

In [16, 17] the Painlevé dynamical system $\text{P}_{\text{VI}}(\kappa)$ is formulated as a holomorphic uniform foliations on the fibration (16) which is compatible with the diagram (18). Thus the Poincaré section (4) is reformulated as a group homomorphism

$$\text{PS}_t(\kappa) : \pi_1(T, t) \rightarrow \text{Aut } \mathcal{M}_t(\kappa), \quad (19)$$

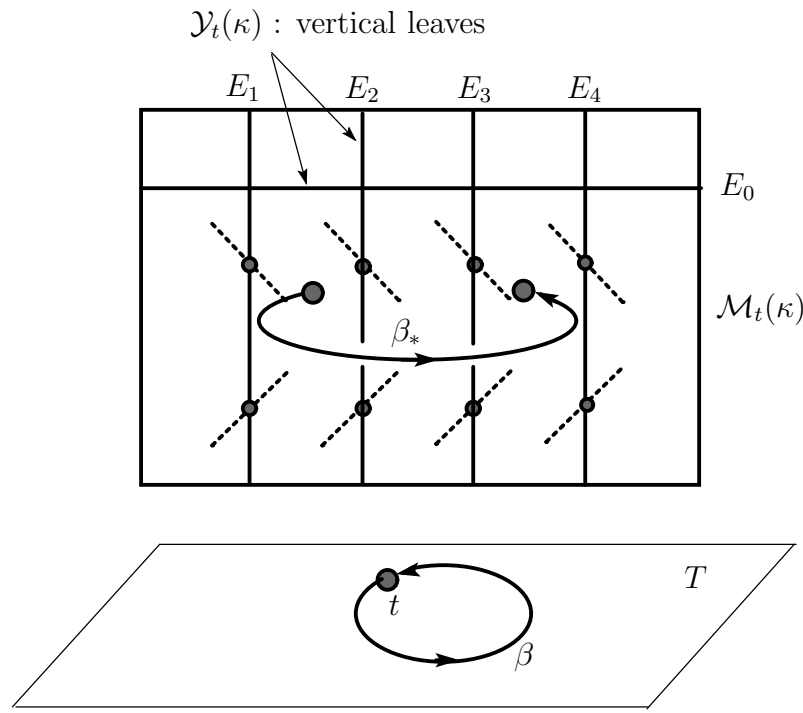


Figure 4: Poincaré section on the space of initial conditions

a visual image of which is given in Figure 4.

Let us describe the fundamental group $\pi_1(T, t)$ in terms of a braid group [3]. We take a base point $t = (t_1, t_2, t_3) \in T$ in such a manner that the three points lie on the real line in an increasing order $t_1 < t_2 < t_3$. To treat them symmetrically, we denote them by t_i, t_j, t_k , where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, and think of them as cyclically ordered three points on the equator $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let β_i be a braid on three strings as in Figure 5 (left) along which t_i and t_j make a half-turn, with t_i moving in the southern hemisphere and t_j in the northern hemisphere, while t_k is kept fixed as in Figure 5 (right). Then the braid group on three strings is the group generated by $\beta_i, \beta_j, \beta_k$, and the pure braid group P_3 is the normal subgroup of B_3 generated by their squares $\beta_i^2, \beta_j^2, \beta_k^2$,

$$P_3 = \langle \beta_i^2, \beta_j^2, \beta_k^2 \rangle \triangleleft B_3 = \langle \beta_i, \beta_j, \beta_k \rangle.$$

The generators of B_3 satisfy relations $\beta_i \beta_j \beta_i = \beta_j \beta_i \beta_j$ and $\beta_k = \beta_i \beta_j \beta_i^{-1}$, so that B_3 is generated by β_i and β_j only. The fundamental group $\pi_1(T, t)$ can be identified with the pure braid group P_3 . The reduction map (17) induces a group homomorphism $r_* : P_3 = \pi_1(T, t) \rightarrow \pi_1(Z, z)$. It is easy to see that this homomorphism sends the three basic pure braids in P_3 to the three basic loops in $\pi_1(Z, z)$ (see Figure 1) in such a manner that

$$r_* : \beta_i^2 \mapsto \gamma_i \quad (i = 1, 2, 3). \quad (20)$$

It is sometimes convenient to lift the Poincaré section (19), which makes sense for pure braids, to the “half-Poincaré section” for ordinary braids. Now let us construct this lift. The symmetric group S_3 acts on T by permuting the entries of $t = (t_1, t_2, t_3)$ and the quotient space T/S_3 is the configuration space of mutually distinct, unordered, three points in \mathbb{C} . The

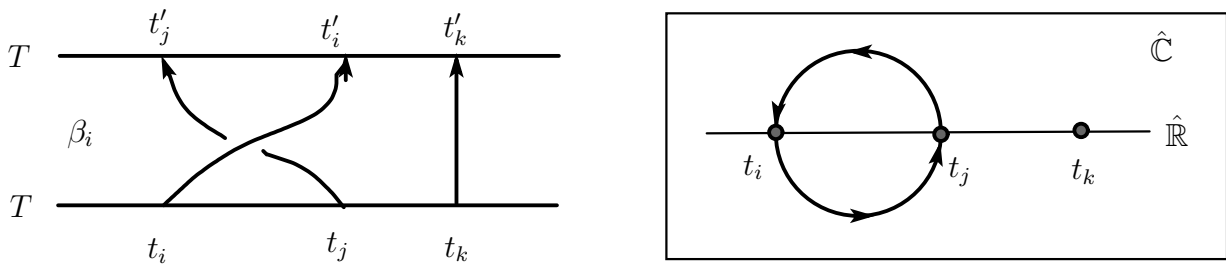


Figure 5: Basic braid β_i in T and the corresponding movement of t in $\hat{\mathbb{C}}$

fundamental group $\pi(T/S_3, s)$ with base point $s = \{t_1, t_2, t_3\}$ is identified with the ordinary braid group B_3 and there exists a short exact sequence of groups

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(T, t) & \longrightarrow & \pi_1(T/S_3, s) & \longrightarrow & S_3 \longrightarrow 1 \\
 & & \parallel & & \parallel & & \parallel \\
 1 & \longrightarrow & P_3 & \longrightarrow & B_3 & \longrightarrow & S_3 \longrightarrow 1.
 \end{array}$$

Then the Poincaré section (19) naturally lifts to a collection of isomorphisms

$$\beta_* : \mathcal{M}_t(\kappa) \rightarrow \mathcal{M}_{\tau(t)}(\tau(\kappa)), \quad (\beta \in B_3)$$

which should be called the *half-Poincaré section* of $P_{VI}(\kappa)$, where $\tau \in S_3$ denotes the permutation corresponding to $\beta \in B_3$. Note that $\tau \in S_3$ acts on $\kappa \in \mathcal{K}$ by permuting the entries of $(\kappa_1, \kappa_2, \kappa_3)$ in the same manner as it does on $t = (t_1, t_2, t_3) \in T$, since κ_i is loaded on t_i . Now the permutation corresponding to the basic braid β_i is the substitution $\tau_i = (i, j)$ that exchanges t_i and t_j while keeping t_k fixed. Thus there are three basic half-Poincaré maps:

$$\beta_{i*} : \mathcal{M}_t(\kappa) \rightarrow \mathcal{M}_{\tau_i(t)}(\tau_i(\kappa)), \quad (i = 1, 2, 3). \quad (21)$$

4 Riemann-Hilbert Correspondence

It is very difficult or rather hopeless to deal with the Painlevé flow directly, since it is a highly transcendental dynamical system on the moduli space of stable parabolic connections. A good idea is to recast it to a more tractable dynamical system, called an isomonodromic flow, on a moduli space of monodromy representations via a Riemann-Hilbert correspondence. We review the construction of such a Riemann-Hilbert correspondence in the sequel.

Let $A := \mathbb{C}^4$ be the complex 4-space with coordinates $a = (a_1, a_2, a_3, a_4)$, called the space of local monodromy data. Given $(t, a) \in T \times A$, let $\mathcal{R}_t(a)$ be the moduli space of Jordan equivalence classes of representations $\rho : \pi_1(\mathbb{P}^1 - D_t, *) \rightarrow SL_2(\mathbb{C})$ such that $\text{Tr } \rho(C_i) = a_i$ for $i \in \{1, 2, 3, 4\}$, where the divisor $D_t = t_1 + t_2 + t_3 + t_4$ is identified with the point set $\{t_1, t_2, t_3, t_4\}$ and C_i is a loop surrounding t_i as in Figure 6. Any stable parabolic connection $Q = (E, \nabla, \psi, l) \in \mathcal{M}_t(\kappa)$, when restricted to $\mathbb{P}^1 - D_t$, induces a flat connection

$$\nabla|_{\mathbb{P}^1 - D_t} : E|_{\mathbb{P}^1 - D_t} \rightarrow (E|_{\mathbb{P}^1 - D_t}) \otimes \Omega_{\mathbb{P}^1 - D_t}^1,$$

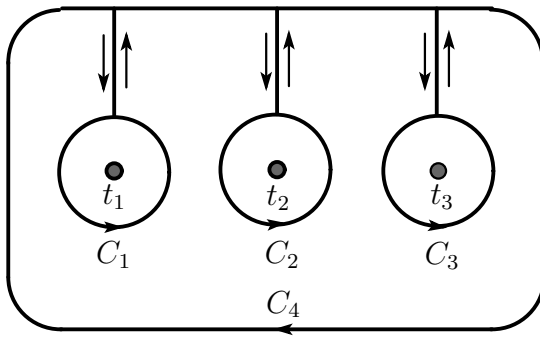


Figure 6: Four loops in $\mathbb{P}^1 - D_t$; the fourth point t_4 is outside C_4 , invisible.

and one can speak of the Jordan equivalence class ρ of its monodromy representations. Then the Riemann-Hilbert correspondence at $t \in T$ is defined by

$$\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \rightarrow \mathcal{R}_t(a), \quad Q \mapsto \rho, \quad (22)$$

where in view of the Riemann scheme in Table 1, the local monodromy data $a \in A$ is given by

$$a_i = \begin{cases} 2 \cos \pi \kappa_i & (i = 1, 2, 3), \\ -2 \cos \pi \kappa_4 & (i = 4). \end{cases} \quad (23)$$

As a relative setting over T , let $\pi_a : \mathcal{R}(a) \rightarrow T$ be the family of moduli spaces of monodromy representations with fiber $\mathcal{R}_t(a)$ over $t \in T$. Then the relative version of Riemann-Hilbert correspondence is formulated as the commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\kappa) & \xrightarrow{\mathrm{RH}_\kappa} & \mathcal{R}(a) \\ \pi_\kappa \downarrow & & \downarrow \pi_a \\ T & \xlongequal{\quad} & T, \end{array} \quad (24)$$

whose fiber over $t \in T$ is given by (22). Then we have the following theorem [16, 17].

Theorem 4.1 *If $\kappa \in \mathcal{K} - \mathbf{Wall}$, then $\mathcal{R}(a)$ as well as each fiber $\mathcal{R}_t(a)$ is smooth and the Riemann-Hilbert correspondence RH_κ in (24) is a biholomorphism.*

Remark 4.2 If $\kappa \in \mathbf{Wall}$, then $\mathcal{R}_t(a)$ is not a smooth surface but a surface with Klein singularities and (22) yields an analytic minimal resolution of singularities, so that (24) gives a family of resolutions of singularities. We refer to [16] for a detailed description of these singularity structures. As is mentioned in Remark 2.11, this fact makes the treatment of the nongeneric case more involved and we leave this case in another occasion.

5 Cubic Surface and the 27 Lines

In this section, following the construction in [16], we shall realize the moduli space $\mathcal{R}_t(a)$ of monodromy representations as an affine cubic surface $\mathcal{S}(\theta)$ and describe the braid group action

on $\mathcal{R}_t(a)$ explicitly in terms of $\mathcal{S}(\theta)$. Moreover we discuss some materials from the geometry of a cubic surface, including the 27 lines on it, as a preliminary to the later sections.

Given $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}_\theta^4$, we consider an affine cubic surface

$$\mathcal{S}(\theta) = \{x = (x_1, x_2, x_3) \in \mathbb{C}_x^3 : f(x, \theta) = 0\},$$

where the cubic polynomial $f(x, \theta)$ of x with parameter θ is given by

$$f(x, \theta) = x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4.$$

Then there exists an isomorphism of affine algebraic surfaces, $\mathcal{R}_t(a) \rightarrow \mathcal{S}(\theta)$, $\rho \mapsto x$, where

$$x_i = \text{Tr } \rho(C_j C_k), \quad \text{for } \{i, j, k\} = \{1, 2, 3\},$$

together with a correspondence of parameters, $A \rightarrow \Theta$, $a \mapsto \theta$, given by

$$\theta_i = \begin{cases} a_i a_4 + a_j a_k & (\{i, j, k\} = \{1, 2, 3\}), \\ a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 & (i = 4). \end{cases} \quad (25)$$

With this identification, the Riemann-Hilbert correspondence (22) is reformulated as a map

$$\text{RH}_t(\kappa) : \mathcal{M}_t(\kappa) \rightarrow \mathcal{S}(\theta), \quad \text{with } \theta = \text{rh}(\kappa), \quad (26)$$

where $\text{rh} : \mathcal{K} \rightarrow \Theta$ is the composition of the maps $\mathcal{K} \rightarrow A$ and $A \rightarrow \Theta$ defined by (23) and (25), and is referred to as the Riemann-Hilbert correspondence in the parameter level. Through the reformulated Riemann-Hilbert correspondence (26), the i -th basic half-Poincaré map β_{i*} in (21) is conjugated to a map $g_i : \mathcal{S}(\theta) \rightarrow \mathcal{S}(\theta')$, $(x, \theta) \mapsto (x', \theta')$, defined by

$$g_i : (x'_i, x'_j, x'_k, \theta'_i, \theta'_j, \theta'_k, \theta'_4) = (\theta_j - x_j - x_k x_i, x_i, x_k, \theta_j, \theta_i, \theta_k, \theta_4), \quad (27)$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. A derivation of this formula can be found in [20] (see also [5, 9, 12, 19, 23]). The map (27) is strictly conjugate to the map (21), since (26) is biholomorphic by Theorem 4.1. We can easily check the relations $g_i g_j g_i = g_j g_i g_j$ and $g_k = g_i g_j g_i^{-1}$, which are just parallel to those for the braids $\beta_i, \beta_j, \beta_k$.

Remark 5.1 The affine cubic surface $\mathcal{S}(\theta)$ admits a natural complex area form

$$\omega(\theta) = \frac{dx_1 \wedge dx_2 \wedge dx_3}{d_x f(x, \theta)}, \quad (28)$$

the Poincaré residue for the surface $\mathcal{S}(\theta)$. The transformations g_i are area-preserving with respect to $\omega(\theta)$. It is known [16, 18, 20] that the standard area form $\omega_t(\kappa)$ on the moduli space $\mathcal{M}_t(\kappa)$ in Remark 3.4 is the pull-back of $\omega(\theta)$ by the Riemann-Hilbert correspondence (26).

In order to utilize standard techniques from algebraic geometry and complex geometry, we need to compactify the affine cubic surface $\mathcal{S}(\theta)$ by a standard embedding

$$\mathcal{S}(\theta) \hookrightarrow \overline{\mathcal{S}}(\theta) \subset \mathbb{P}^3, \quad x = (x_1, x_2, x_3) \mapsto [1 : x_1 : x_2 : x_3],$$

where the compactified surface $\overline{\mathcal{S}}(\theta)$ is defined by $\overline{\mathcal{S}}(\theta) = \{X \in \mathbb{P}^3 : F(X, \theta) = 0\}$ with

$$F(X, \theta) = X_1 X_2 X_3 + X_0(X_1^2 + X_2^2 + X_3^2) - X_0^2(\theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3) + \theta_4 X_0^3.$$

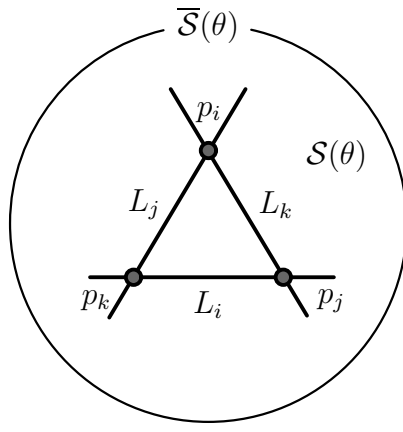


Figure 7: Tritangent lines at infinity on $\overline{\mathcal{S}}(\theta)$

It is obtained from the affine surface $\mathcal{S}(\theta)$ by adding three lines at infinity,

$$L_i = \{ X \in \mathbb{P}^3 : X_0 = X_i = 0 \} \quad (i = 1, 2, 3). \quad (29)$$

The union $L = L_1 \cup L_2 \cup L_3$ is called the tritangent lines at infinity and the intersection point of L_j and L_k is denoted by p_i as in Figure 7. Note that

$$p_1 = [0 : 1 : 0 : 0], \quad p_2 = [0 : 0 : 1 : 0], \quad p_3 = [0 : 0 : 0 : 1].$$

For $i \in \{1, 2, 3\}$ we put $U_i = \{ X \in \mathbb{P}^3 : X_i \neq 0 \}$ and take inhomogeneous coordinates of \mathbb{P}^3 as

$$\begin{aligned} u &= (u_0, u_j, u_k) = (X_0/X_i, X_j/X_i, X_k/X_i) && \text{on } U_i, \\ v &= (v_0, v_i, v_k) = (X_0/X_j, X_i/X_j, X_k/X_j) && \text{on } U_j, \\ w &= (w_0, w_i, w_j) = (X_0/X_k, X_i/X_k, X_j/X_k) && \text{on } U_k, \end{aligned} \quad (30)$$

where $\{i, j, k\} = \{1, 2, 3\}$. In terms of these coordinates we shall find local coordinates and local equations of $\overline{\mathcal{S}}(\theta)$ around L . Since $L \subset U_1 \cup U_2 \cup U_3$, we can divide L into three components $L \cap U_i$, $i = 1, 2, 3$, and make a further decomposition

$$L \cap U_i = \{p_i\} \cup (L_j - \{p_i, p_k\}) \cup (L_k - \{p_i, p_j\}) \quad (\{i, j, k\} = \{1, 2, 3\})$$

into a total of nine pieces. Then a careful inspection of equation $F(X, \theta) = 0$ implies that around those pieces we can take local coordinates and local equations as in Table 2, where $O_m(u_j, u_k) = O(|u_j| + |u_k|)^m$ denotes a small term of order m as $(u_j, u_k) \rightarrow (0, 0)$.

Lemma 5.2 *As to the smoothness of the surface $\overline{\mathcal{S}}(\theta)$, the following hold.*

- (1) *For any $\theta \in \Theta$, the surface $\overline{\mathcal{S}}(\theta)$ is smooth in a neighborhood of L .*
- (2) *If $\theta = \text{rh}(\kappa)$ with $\kappa \in \mathcal{K}$, the surface $\overline{\mathcal{S}}(\theta)$ is smooth everywhere if and only if $\kappa \in \mathcal{K} - \mathbf{Wall}$.*

Proof. In terms of the inhomogeneous coordinates u in (30), we have

$$\overline{\mathcal{S}}(\theta) \cap U_i \cong \{ u = (u_0, u_j, u_k) \in \mathbb{C}^3 : f_i(u, \theta) = 0 \},$$

coordinates	valid around	local equation
(u_j, u_k)	p_i	$u_0 = -(u_j u_k) \{1 - (u_j^2 + \theta_i u_j u_k + u_k^2) + O_3(u_j, u_k)\}$
(u_0, u_k)	$L_j - \{p_i, p_k\}$	$u_j = -(u_k + 1/u_k)u_0 + (\theta_k + \theta_i/u_k)u_0^2 + O(u_0^3)$
(u_0, u_j)	$L_k - \{p_i, p_j\}$	$u_k = -(u_j + 1/u_j)u_0 + (\theta_j + \theta_i/u_j)u_0^2 + O(u_0^3)$
(v_i, v_k)	p_j	$v_0 = -(v_i v_k) \{1 - (v_i^2 + \theta_j v_i v_k + v_k^2) + O_3(v_i, v_k)\}$
(v_0, v_i)	$L_k - \{p_i, p_j\}$	$v_k = -(v_i + 1/v_i)v_0 + (\theta_i + \theta_j/v_i)v_0^2 + O(v_0^3)$
(v_0, v_k)	$L_i - \{p_j, p_k\}$	$v_i = -(v_k + 1/v_k)v_0 + (\theta_k + \theta_j/v_k)v_0^2 + O(v_0^3)$
(w_i, w_j)	p_k	$w_0 = -(w_i w_j) \{1 - (w_i^2 + \theta_k w_i w_j + w_j^2) + O_3(w_i, w_j)\}$
(w_0, w_j)	$L_i - \{p_j, p_k\}$	$w_i = -(w_j + 1/w_j)w_0 + (\theta_j + \theta_k/w_j)w_0^2 + O(w_0^3)$
(w_0, w_i)	$L_j - \{p_i, p_k\}$	$w_j = -(w_i + 1/w_i)w_0 + (\theta_i + \theta_k/w_i)w_0^2 + O(w_0^3)$

Table 2: Local coordinates and local equations of $\overline{\mathcal{S}}(\theta)$

where the defining equation $f_i(u, \theta)$ is given by

$$f_i(u, \theta) = u_j u_k + u_0(1 + u_j^2 + u_k^2) - u_0^2(\theta_i + \theta_j u_j + \theta_k u_k) + \theta_4 u_0^3.$$

The partial derivatives of $f_i = f_i(u, \theta)$ with respect to $u = (u_0, u_j, u_k)$ are calculated as

$$\begin{aligned} \frac{\partial f_i}{\partial u_0} &= (1 + u_j^2 + u_k^2) - 2u_0(\theta_i + \theta_j u_j + \theta_k u_k) + 3\theta_4 u_0^2 \\ \frac{\partial f_i}{\partial u_j} &= u_k + 2u_0 u_j - \theta_j u_0^2 \\ \frac{\partial f_i}{\partial u_k} &= u_j + 2u_0 u_k - \theta_k u_0^2. \end{aligned}$$

Restricted to the set $L \cap U_i = (L_j \cap U_i) \cup (L_k \cap U_i)$, these derivatives become

$$\begin{aligned} \frac{\partial f_i}{\partial u_0} &= 1 + u_k^2, & \frac{\partial f_i}{\partial u_j} &= u_k, & \frac{\partial f_i}{\partial u_k} &= 0, & \text{on } L_j \cap U_i, \\ \frac{\partial f_i}{\partial u_0} &= 1 + u_j^2, & \frac{\partial f_i}{\partial u_j} &= 0, & \frac{\partial f_i}{\partial u_k} &= u_j, & \text{on } L_k \cap U_i. \end{aligned}$$

Hence the exterior derivative $d_u f_i$ does not vanish on $L \cap U_i$, and the implicit function theorem implies that $\overline{\mathcal{S}}(\theta)$ is smooth in a neighborhood of L . This proves assertion (1). In order to show assertion (2) we recall that the affine surface $\mathcal{S}(\theta)$ is smooth if and only if $\theta = \text{rh}(\kappa)$ with $\kappa \in \mathcal{K} - \mathbf{Wall}$ (see [16]). Then assertion (2) readily follows from assertion (1). \square

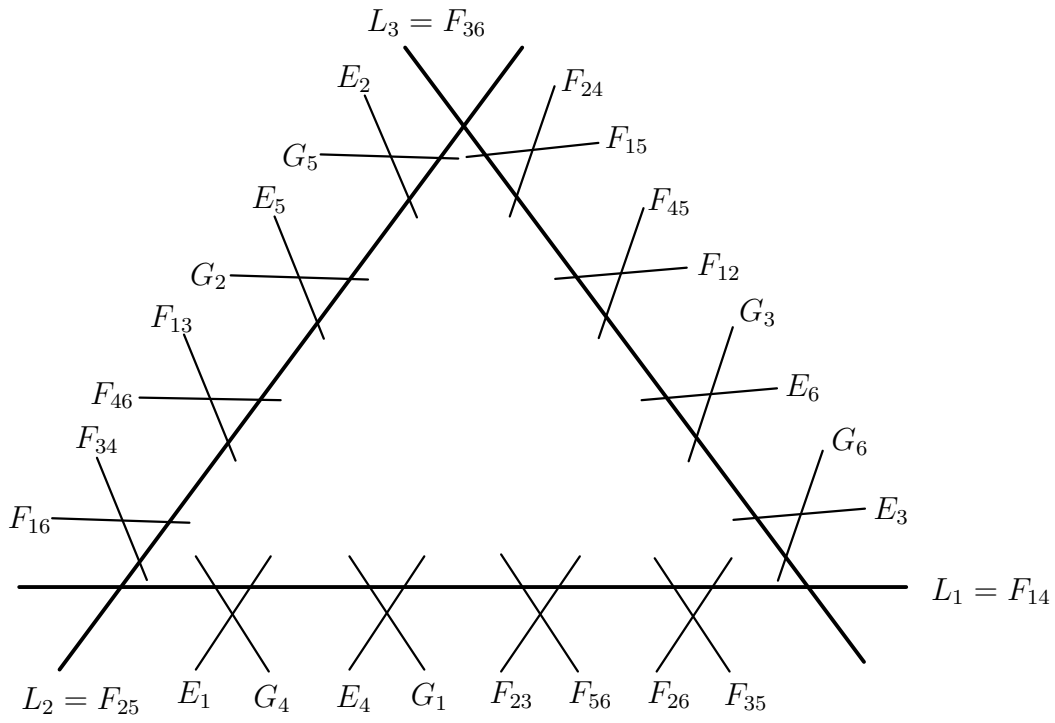


Figure 8: The 27 lines on $\overline{\mathcal{S}}(\theta)$ viewed from the tritangent plane at infinity

Now let us review some basic facts about smooth cubic surfaces in \mathbb{P}^3 (see e.g. [11]). It is well known that every smooth cubic surface S in \mathbb{P}^3 can be obtained by blowing up \mathbb{P}^2 at six points P_1, \dots, P_6 , no three colinear and not all six on a conic, and embedding the blow-up surface into \mathbb{P}^3 by the proper transform of the linear system of cubics passing through the six points P_1, \dots, P_6 . It is also well known that there are exactly 27 lines on the smooth cubic surface S , each of which has self-intersection number -1 . Explicitly, they are given by

$$E_a \quad (a = 1, \dots, 6); \quad F_{ab} \quad (1 \leq a < b \leq 6); \quad G_a \quad (a = 1, \dots, 6),$$

- (1) E_a is the exceptional curve over the point P_a ,
- (2) F_{ab} is the strict transform of the line in \mathbb{P}^2 through the two points P_a and P_b ,
- (3) G_a is the strict transform of the conic in \mathbb{P}^2 through the five points $P_1, \dots, \hat{P}_a, \dots, P_6$.

Here the index a should not be confused with the local monodromy data $a \in A$. All the intersection relations among the 27 lines with *nonzero* intersection numbers are listed as

$$\begin{aligned} (E_a, E_a) &= (G_a, G_a) = (F_{ab}, F_{ab}) = -1 & (\forall a, b), \\ (E_a, F_{bc}) &= (G_a, F_{bc}) = 1 & (a \in \{b, c\}), \\ (E_a, G_b) &= 1 & (a \neq b), \\ (F_{ab}, F_{cd}) &= 1 & (\{a, b\} \cap \{c, d\} = \emptyset). \end{aligned}$$

Moreover there are exactly 45 tritangent planes that cut out a triplet of lines on S . In our case $S = \overline{\mathcal{S}}(\theta)$, the plane at infinity $\{X \in \mathbb{P}^3 : X_0 = 0\}$ is an instance of tritangent plane,

1	$L_i(b_i, b_4; b_j, b_k)$	$L_i(1/b_i, 1/b_4; b_j, b_k)$
2	$L_i(b_j, b_k; b_i, b_4)$	$L_i(1/b_j, 1/b_k; b_i, b_4)$
3	$L_i(1/b_i, b_4; b_j, b_k)$	$L_i(b_i, 1/b_4; b_j, b_k)$
4	$L_i(1/b_j, b_k; b_i, b_4)$	$L_i(b_j, 1/b_k; b_i, b_4)$

Table 3: Eight lines intersecting the line L_i at infinity, divided into four pairs

which cuts out the lines in (29). The arrangement of the 27 lines viewed from the tritangent plane at infinity is shown in Figure 8 and the lines at infinity are given by

$$L_1 = F_{14}, \quad L_2 = F_{25}, \quad L_3 = F_{36}. \quad (31)$$

Each line at infinity is intersected by exactly eight lines and this fact enables us to divide the 27 lines into three groups of nine lines labeled by lines at infinity. *Caution:* only the intersection relations among the lines of the same group are indicated and no other intersection relations are depicted in Figure 8.

If E_0 is the strict transform of a line in \mathbb{P}^2 not passing through P_1, \dots, P_6 relative to the 6-point blow-up $S \rightarrow \mathbb{P}^2$, then the second cohomology group of $S = \overline{\mathcal{S}}(\theta)$ is expressed as

$$H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) = \mathbb{Z}E_0 \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}E_4 \oplus \mathbb{Z}E_5 \oplus \mathbb{Z}E_6, \quad (32)$$

where a divisor is identified with the cohomology class it represents. It is a Lorentzian lattice of rank 7 with intersection numbers

$$(E_a, E_b) = \begin{cases} 1 & (a = b = 0), \\ -1 & (a = b \neq 0), \\ 0 & (\text{otherwise}). \end{cases} \quad (33)$$

In terms of the basis in (32) the lines F_{ab} and G_a are represented as

$$F_{ab} = E_0 - E_a - E_b, \quad G_a = 2E_0 - (E_1 + \dots + \widehat{E}_a + \dots + E_6). \quad (34)$$

We shall describe the 27 lines on our cubic surface $\overline{\mathcal{S}}(\theta)$ under the condition that $\overline{\mathcal{S}}(\theta)$ is smooth, namely, $\theta = \text{rh}(\kappa)$ with $\kappa \in \mathcal{K} - \mathbf{Wall}$. To this end we introduce new parameters $b = (b_1, b_2, b_3, b_4) \in B := (\mathbb{C}^\times)^4$ in such a manner that b is expressed as

$$b_i = \begin{cases} \exp(\sqrt{-1}\pi\kappa_i) & (i = 1, 2, 3), \\ -\exp(\sqrt{-1}\pi\kappa_4) & (i = 4), \end{cases}$$

as a function of $\kappa \in \mathcal{K}$. Then the Riemann scheme in Table 1 implies that b_i is an eigenvalue of the monodromy matrix $\rho(C_i)$ around the point t_i and formula (23) implies that $a_i = b_i + b_i^{-1}$. Here parameters $b \in B$ should not be confused with the index b above. In terms of the parameters $b \in B$, the discriminant $\Delta(\theta)$ of the cubic surfaces $\mathcal{S}(\theta)$ factors as

$$\Delta(\theta) = \prod_{l=1}^4 (b_l - b_l^{-1})^2 \prod_{\varepsilon \in \{\pm 1\}^4} (b^\varepsilon - 1), \quad (35)$$

where $b^\varepsilon = b_1^{\varepsilon_1} b_2^{\varepsilon_2} b_3^{\varepsilon_3} b_4^{\varepsilon_4}$ for each quadruple sign $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{\pm 1\}^4$. Formula (35) clearly shows for which parameters $b \in B$ the cubic surface $\overline{\mathcal{S}}(\theta)$ is smooth or singular.

Assume that $\overline{\mathcal{S}}(\theta)$ is smooth, namely, $\Delta(\theta) \neq 0$. Then, as is mentioned earlier, for each index $i \in \{1, 2, 3\}$ with $\{i, j, k\} = \{1, 2, 3\}$, there are exactly eight lines on $\overline{\mathcal{S}}(\theta)$ intersecting L_i , but not intersecting the remaining two lines at infinity, L_j and L_k . They are just $\{E_i, G_{i+3}\}$, $\{E_{i+3}, G_i\}$, $\{F_{jk}, F_{j+3, k+3}\}$, $\{F_{j, k+3}, F_{j+3, k}\}$ as in Figure 8, where two lines from the same pair intersect, but ones from different pairs are disjoint. In terms of the parameters $b \in B$ introduced above, those eight lines are given as in Table 3, where $L_i(b_i, b_4; b_j, b_k)$ stands for the line in \mathbb{P}^3 defined by the system of linear equations

$$X_i = (b_i b_4 + b_i^{-1} b_4^{-1}) X_0, \quad X_j + (b_i b_4) X_k = \{b_i(b_k + b_k^{-1}) + b_4(b_j + b_j^{-1})\} X_0. \quad (36)$$

6 Dynamical System on Cubic Surface

The affine cubic surface $\mathcal{S}(\theta)$ is a $(2, 2, 2)$ -surface, that is, its defining equation $f(x, \theta) = 0$ is a quadratic equation in each variable x_i , $i = 1, 2, 3$. Therefore the line through a point $x \in \mathcal{S}(\theta)$ parallel to the x_i -axis passes through a unique second point $x' \in \mathcal{S}(\theta)$ (see Figure 9). This defines an involution $\sigma_i : \mathcal{S}(\theta) \rightarrow \mathcal{S}(\theta)$, $x \mapsto x'$, which is explicitly given by

$$\sigma_i : (x'_i, x'_j, x'_k) = (\theta_i - x_i - x_j x_k, x_j, x_k), \quad (i = 1, 2, 3). \quad (37)$$

It is easy to see that the involution σ_i preserves the Poincaré residue $\omega(\theta)$ in (28).

The automorphism σ_i of the affine surface $\mathcal{S}(\theta)$ extends to a birational map of the projective surface $\overline{\mathcal{S}}(\theta)$, which will also be denoted by σ_i . In terms of the homogeneous coordinates X of \mathbb{P}^3 , the birational map $\sigma_i : X \mapsto X'$ is expressed as

$$[X'_0 : X'_i : X'_j : X'_k] = [X_0^2 : \theta_i X_0^2 - X_0 X_i - X_j X_k : X_0 X_j : X_0 X_k].$$

Let $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ be the group of birational transformations on $\overline{\mathcal{S}}(\theta)$ generated by the involutions $\sigma_1, \sigma_2, \sigma_3$. It will turn out that G is a universal Coxeter group of rank three with generators $\sigma_1, \sigma_2, \sigma_3$ (see Theorem 7.7). We are interested in the dynamics of the G -action on $\overline{\mathcal{S}}(\theta)$. Usually the dynamics of a group action is more involved than that of a single transformation; more techniques and tools have been developed for the latter rather than for the former. So in this article we pick up each individual transformation from the group G and study its dynamics, leaving the interaction of plural transformations in another occasion.

In order to study the dynamics of any element $\sigma \in G$, we begin with investigating the basic elements σ_i , $i = 1, 2, 3$, especially their behaviors in a neighborhood of the tritangent lines L at infinity. To this end let us introduce the following three points

$$q_1 = [0 : 0 : 1 : 1], \quad q_2 = [0 : 1 : 0 : 1], \quad q_3 = [0 : 1 : 1 : 0],$$

where q_i may be thought of as the “mid-point” of p_j and p_k on the line L_i .

Lemma 6.1 *The birational map σ_i has the following properties (see Figure 10).*

- (1) σ_i blows down the line L_i to the point p_i ,
- (2) σ_i restricts to the automorphism of L_j that fixes q_j and exchanges p_i and p_k ,

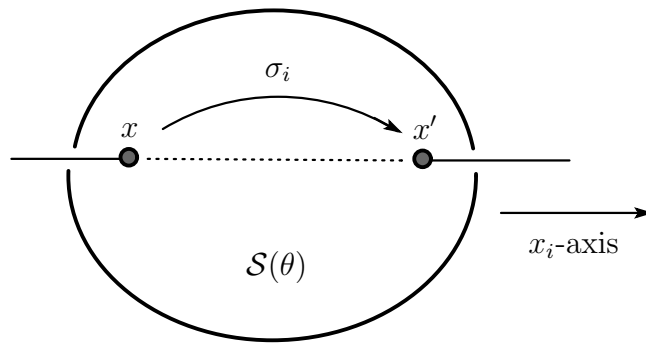


Figure 9: Involutions of a $(2, 2, 2)$ -surface

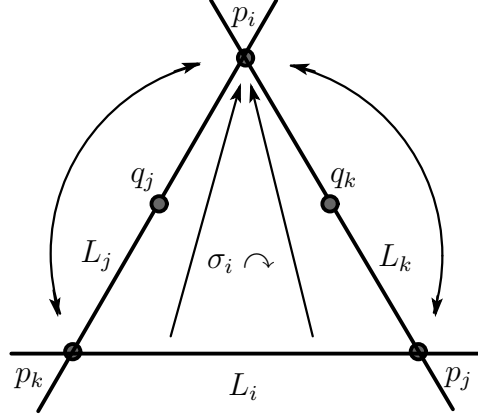


Figure 10: The birational map σ_i restricted to L

- (3) σ_i restricts to the automorphism of L_k that fixes q_k and exchanges p_i and p_j ,
- (4) p_i is the unique indeterminacy point of σ_i ,

Proof. In order to investigate σ_i , we use the inhomogeneous coordinates of \mathbb{P}^3 in (30) and local coordinates and local equations of $\overline{\mathcal{S}}(\theta)$ in Table 2, with target coordinates being dashed.

In terms of the inhomogeneous coordinates v and u' of \mathbb{P}^3 , the map $\sigma_i : v \mapsto u'$ is given by

$$u'_0 = \frac{v_0^2}{\theta_i v_0^2 - v_0 v_i - v_k}, \quad u'_j = \frac{v_0}{\theta_i v_0^2 - v_0 v_i - v_k}, \quad u'_k = \frac{v_0 v_k}{\theta_i v_0^2 - v_0 v_i - v_k}. \quad (38)$$

In a neighborhood of $L_i - \{p_j, p_k\}$ in $\overline{\mathcal{S}}(\theta)$, using $v_i = O(v_0)$, we observe that

$$\theta_i v_0^2 - v_0 v_i - v_k = -v_k \{1 + O(v_0^2)\},$$

which is substituted into (38) to yield

$$u'_j = -\frac{v_0}{v_k \{1 + O(v_0^2)\}} = -\frac{v_0}{v_k} \{1 + O(v_0^2)\}, \quad u'_k = -\frac{v_0 v_k}{v_k \{1 + O(v_0^2)\}} = -v_0 \{1 + O(v_0^2)\}.$$

In particular putting $v_0 = 0$ leads to $u'_j = u'_k = 0$. This means that σ_i maps a neighborhood of $L_i - \{p_j, p_k\}$ to a neighborhood of p_i , collapsing $L_i - \{p_j, p_k\}$ to the single point p_i .

In a similar manner, in a neighborhood of p_j in $\overline{\mathcal{S}}(\theta)$ we observe that

$$v_0 = -(v_i v_k) \{1 + O_2(v_i, v_k)\}, \quad \theta_i v_0^2 - v_0 v_i - v_k = -v_k \{1 + O_2(v_i, v_k)\},$$

which are substituted into (38) to yield

$$u'_j = v_i \{1 + O_2(v_i, v_k)\}, \quad u'_k = (v_i v_k) \{1 + O_2(v_i, v_k)\}.$$

In particular putting $v_i = 0$ leads to $u'_j = u'_k = 0$. This means that σ_i maps a neighborhood of p_j to a neighborhood of p_i , collapsing a neighborhood in L_i of p_j to the single point p_i . Using (w, u') in place of (v, u') , we can argue similarly in a neighborhood of p_k . Therefore σ_i blows down L_i to the point p_i , which proves assertion (1). Moreover it is clear from the argument that there is no indeterminacy point on the line L_i .

In terms of the inhomogeneous coordinates u and u' of \mathbb{P}^3 the map $\sigma_i : u \mapsto u'$ is given by

$$u'_0 = \frac{u_0^2}{\theta_i u_0^2 - u_0 - u_j u_k}, \quad u'_j = \frac{u_0 u_j}{\theta_i u_0^2 - u_0 - u_j u_k}, \quad u'_k = \frac{u_0 u_k}{\theta_i u_0^2 - u_0 - u_j u_k}. \quad (39)$$

In a neighborhood of $L_j - \{p_i, p_k\}$ in $\overline{\mathcal{S}}(\theta)$, using $u_j = -(u_k + 1/u_k)u_0 + O(u_0^2)$, we have

$$\theta_i u_0^2 - u_0 - u_j u_k = u_0 \{u_k^2 + O(u_0)\},$$

which is substituted into (39) to yield

$$u'_0 = \frac{u_0}{u_k^2 + O(u_0)} = \frac{u_0}{u_k^2} + O(u_0^2), \quad u'_k = \frac{u_k}{u_k^2 + O(u_0)} = \frac{1}{u_k} + O(u_0).$$

In particular putting $u_0 = 0$ leads to $u'_0 = 0$ and $u'_k = 1/u_k$. This means that σ_i restricts to an automorphism of a neighborhood of $L_j - \{p_i, p_k\}$ in $\overline{\mathcal{S}}(\theta)$ which induces a unique automorphism of L_i fixing q_j and exchanging p_i and p_k . This proves assertion (2) and also shows that there is no indeterminacy point on $L_j - \{p_i, p_k\}$. Assertion (3) and the nonexistence of indeterminacy point on $L_k - \{p_i, p_j\}$ are established just in the same manner.

From the above argument we have already known that there is no indeterminacy point other than p_i . Then the point p_i is actually an indeterminacy point, because σ_i is an involution blowing down L_i to p_i and hence blows up p_i to L_i reciprocally. This proves assertion (4). \square

Later we will need some information about how the involution σ_i transforms a line to another curve, which is stated in the following lemma.

Lemma 6.2 *For any $\{i, j, k\} = \{1, 2, 3\}$, the involution σ_i satisfies the following properties:*

- (1) $\sigma_i(E_i)$ intersects E_i at two points counted with multiplicity. Similarly, $\sigma_i(E_{i+3})$ intersects E_{i+3} at two points counted with multiplicity.
- (2) $\sigma_i(E_i)$ intersects E_{i+3} at one point counted with multiplicity. Similarly, $\sigma_i(E_{i+3})$ intersects E_i at one point counted with multiplicity.
- (3) σ_i exchanges the lines E_j and G_{j+3} ; E_{j+3} and G_j ; E_k and G_{k+3} ; E_{k+3} and G_k , respectively.

Proof. By Table 3 we may put $E_i = L_i(b_i, b_4; b_j, b_k)$ and $E_{i+3} = L_i(b_j, b_k; b_i, b_4)$. Assertion (1) of Lemma 6.1 implies that $\sigma_i(E_i)$ does not intersect E_i nor E_{i+3} at any point at infinity. So we can work with the inhomogeneous coordinates $x = (x_1, x_2, x_3)$. By (36) the line E_i is given by

$$x_i = b_i b_4 + (b_i b_4)^{-1}, \quad x_j + (b_i b_4) x_k = a_k b_i + a_j b_4. \quad (40)$$

In a similar manner, by exchanging (b_i, b_4) and (b_j, b_k) in (36), the line E_{i+3} is given by

$$x_i = b_j b_k + (b_j b_k)^{-1}, \quad x_j + (b_j b_k) x_k = a_4 b_j + a_i b_k. \quad (41)$$

Moreover, by applying formula (37) to (40), the curve $\sigma_i(E_i)$ is expressed as

$$\theta_i - x_i - x_j x_k = b_i b_4 + (b_i b_4)^{-1}, \quad x_j + (b_i b_4) x_k = a_k b_i + a_j b_4. \quad (42)$$

Note that the second equations of (40) and (42) are the same.

In order to find out the intersection of $\sigma_i(E_i)$ with E_i , let us couple (40) and (42) together. Eliminating x_i and x_j we obtain a quadratic equation for x_k ,

$$(b_i b_4) x_k^2 - (a_k b_i + a_j b_4) x_k + \theta_i - 2\{b_i b_4 + (b_i b_4)^{-1}\} = 0.$$

For a simple root of this equation we have a simple intersection point of $\sigma_i(E_i)$ with E_i and for a double root we have an intersection point of multiplicity two. This proves assertion (1) for the pair $\sigma_i(E_i)$ and E_i . The assertion (1) for $\sigma_i(E_{i+3})$ and E_{i+3} is proved in a similar manner.

Next, in order to find out the intersection of $\sigma_i(E_i)$ with E_{i+3} , let us couple (41) and (42). From the first equation of (41) the x_i -coordinate is already fixed. The second equations of (41) and (42) are coupled to yield a linear system for x_j and x_k , whose determinant

$$b_j b_k - b_i b_4 = b_i b_4 (b_i^{-1} b_j b_k b_4^{-1} - 1)$$

is nonzero by the assumption that $\overline{\mathcal{S}}(\theta)$ is smooth, that is, the discriminant $\Delta(\theta)$ in (35) is nonzero. Then the linear system is uniquely solved to determine x_j and x_k . Now we can check that the first equation of (42) is redundant, that is, automatically satisfied. Therefore $\sigma_i(E_i)$ and E_{i+3} has a simple intersection, which implies assertion (2) for the pair $\sigma_i(E_i)$ and E_{i+3} . The assertion (2) for $\sigma_i(E_{i+3})$ and E_i is proved in a similar manner.

Finally we see that σ_i exchanges E_j and G_{j+3} . We may put $E_j = L_j(b_j, b_4; b_k, b_i)$ and $G_{j+3} = L_j(1/b_j, 1/b_4; b_k, b_i)$. By formula (36) (with indices suitably permuted), these lines are given by

$$x_j = b_j b_4 + (b_j b_4)^{-1}, \quad x_k + (b_j b_4) x_i = a_i b_j + a_k b_4, \quad (43)$$

$$x_j = b_j b_4 + (b_j b_4)^{-1}, \quad x_k + (b_j b_4)^{-1} x_i = a_i b_j^{-1} + a_k b_4^{-1}. \quad (44)$$

Using formula (37) we can check that equations (43) and (44) are transformed to each other by σ_i . This together with similar argument for the other lines establishes assertion (3). \square

7 Cohomological Action

A general theory of the dynamical system for a bimeromorphic map of a surface is developed in [7]. The basic strategy employed there is to consider the induced action of the map on the $(1, 1)$ -cohomology group, taking into account the influence of its exceptional set and indeterminacy set. In this section we shall use this technique in our context.

Let S be a compact complex surface, $f : S \dashrightarrow S$ a bimeromorphic map. Then f is represented by a compact complex surface Γ , called the desingularized graph of f , together with proper modifications $\pi_1 : \Gamma \rightarrow S$ and $\pi_2 : \Gamma \rightarrow S$ such that $f = \pi_2 \circ \pi_1^{-1}$ on a dense open subset. For $i = 1, 2$, let $\mathcal{E}(\pi_i) := \{x \in \Gamma : \# \pi_i^{-1}(\pi_i(x)) = \infty\}$ be the exceptional set for the projection π_i . The images $\mathcal{E}(f) := \pi_1(\mathcal{E}(\pi_2))$ and $I(f) := \pi_1(\mathcal{E}(\pi_1))$ are called the exceptional set and the indeterminacy set of f respectively. Between these sets there is a useful relation

$$f(\mathcal{E}(f)) = I(f^{-1}). \quad (45)$$

If $S = \overline{\mathcal{S}}(\theta)$ and $f = \sigma_i$, then Lemma 6.1 readily leads to the following lemma.

Lemma 7.1 *For each $i \in \{1, 2, 3\}$, we have $\mathcal{E}(\sigma_i) = L_i$, $\sigma_i(\mathcal{E}(\sigma_i)) = \{p_i\}$ and $I(\sigma_i) = \{p_i\}$.*

Given any element $\sigma \in G$ other than the unit element, we can write

$$\sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n}, \quad (46)$$

for some $n \in \mathbb{N}$ and some n -tuple of indices $(i_1, \dots, i_n) \in \{1, 2, 3\}^n$ such that every neighboring indices i_ν and $i_{\nu+1}$ are distinct. It is not yet clear at this stage whether the expression (46) is unique or not, though the uniqueness will be established later (see Theorem 7.7). In any case, we begin with the determination of the exceptional set and the indeterminacy set of σ .

Lemma 7.2 *For the expression (46) we have*

$$\mathcal{E}(\sigma) = \bigcup_{\nu=1}^n L_{i_\nu}, \quad \sigma(\mathcal{E}(\sigma)) = \{p_{i_1}\}, \quad I(\sigma) = \{p_{i_n}\}. \quad (47)$$

Proof. Let us prove the first formula of (47) by induction on the length n . For $n = 1$ the assertion immediately follows from Lemma 7.1. Assume that the assertion holds when the length is $n - 1$ and consider the element $\sigma' = \sigma_{i_2} \cdots \sigma_{i_n}$ of length $n - 1$. Since p_{i_1} and p_{i_2} are distinct, we have $I(\sigma_{i_1}) \cap \sigma'(\mathcal{E}(\sigma')) = \{p_{i_1}\} \cap \{p_{i_2}\} = \emptyset$ and hence $\mathcal{E}(\sigma') \subset \mathcal{E}(\sigma)$. Therefore,

$$\bigcup_{\nu=2}^n L_{i_\nu} \subset \mathcal{E}(\sigma) \subset \bigcup_{\nu=1}^n L_{i_\nu}, \quad (48)$$

where the first inclusion follows from the induction hypothesis and the second inclusion is easily seen from Lemma 6.1. If $i_1 \in \{i_2, \dots, i_n\}$, then the leftmost and rightmost sets in (48) are the same and hence all the three coincide. If $i_1 \notin \{i_2, \dots, i_n\}$, then Lemma 6.1 implies that σ' maps L_{i_1} isomorphically onto itself and then σ_{i_1} blows down L_{i_1} to the single point p_{i_1} . This means that $L_{i_1} \subset \mathcal{E}(\sigma)$ and hence the second inclusion in (48) becomes equality. Thus the assertion is verified for length n and the induction is complete.

The second formula in (47) is also proved by induction on the length n . For $n = 1$ the assertion immediately follows from Lemma 7.1. Assume that the assertion holds when the length is $n - 1$. Then we have $\sigma'(\mathcal{E}(\sigma')) = \{p_{i_2}\}$ by induction hypothesis and hence $\sigma(\mathcal{E}(\sigma)) = \sigma_{i_1}(\mathcal{E}(\sigma_{i_1}) \cup \sigma'(\mathcal{E}(\sigma'))) = \sigma_{i_1}(L_{i_1} \cup \{p_{i_2}\}) = \sigma_{i_1}(L_{i_1}) = \{p_{i_1}\}$, since $p_{i_2} \in L_{i_1}$. This shows that the assertion is verified for length n and hence the induction is complete.

Next we prove the last formula of (47). Instead of σ we consider its inverse σ^{-1} . Since $\sigma^{-1} = \sigma_{i_n} \cdots \sigma_{i_2} \sigma_{i_1}$, the second formula of (47) yields $\sigma^{-1}(\mathcal{E}(\sigma^{-1})) = \{p_{i_n}\}$. Then applying formula (45) to $f = \sigma^{-1}$, we have $I(\sigma) = \{p_{i_n}\}$. Thus the lemma is established. \square

$$\begin{aligned}
\sigma_1^* &= \begin{pmatrix} 6 & 3 & 2 & 2 & 3 & 2 & 2 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \end{pmatrix} & \sigma_2^* &= \begin{pmatrix} 6 & 2 & 3 & 2 & 2 & 3 & 2 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -2 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \end{pmatrix} \\
\sigma_3^* &= \begin{pmatrix} 6 & 2 & 2 & 3 & 2 & 2 & 3 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ -3 & -1 & -1 & -2 & -1 & -1 & -1 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix} & c^* &= \begin{pmatrix} 12 & 6 & 4 & 3 & 6 & 4 & 3 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 \\ -4 & -2 & -2 & -1 & -2 & -1 & -1 \\ -6 & -3 & -2 & -2 & -3 & -2 & -1 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 \\ -4 & -2 & -1 & -1 & -2 & -2 & -1 \\ -6 & -3 & -2 & -1 & -3 & -2 & -2 \end{pmatrix}
\end{aligned}$$

Table 4: Matrix representations of $\sigma_1^*, \sigma_2^*, \sigma_3^*, c^* : H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \curvearrowright$, where $c = \sigma_1 \sigma_2 \sigma_3$

If S is a Kähler surface, two natural actions of f , pull-back and push-forward, on the Dolbeault cohomology group $H^{1,1}(S)$ are defined in the following manner: A smooth $(1,1)$ -form ω on S can be pulled back as a smooth $(1,1)$ -form $\pi_2^* \omega$ on Γ and then pushed forward as a $(1,1)$ -current $\pi_{1*} \pi_2^* \omega$ on S . Hence we define the pull-back $f^* \omega := \pi_{1*} \pi_2^* \omega$ and also the push-forward $f_* \omega = (f^{-1})^* \omega := \pi_{2*} \pi_1^* \omega$. The operators f^* and f_* commute with the exterior differential d and the complex structure of S and so descend to linear actions on $H^{1,1}(S)$. For general bimeromorphic maps f and g , the composition rule $(f \circ g)^* = g^* \circ f^*$ is not necessarily true. However a useful criterion under which this rule becomes true is given in [7].

Lemma 7.3 *If $I(f) \cap f(\mathcal{E}(g)) = \emptyset$, then $(f \circ g)^* = g^* \circ f^* : H^{1,1}(S) \curvearrowright$.*

We shall apply this lemma to our birational transformation σ in (46).

Lemma 7.4 *For the expression (46) we have $\sigma^* = \sigma_{i_n}^* \cdots \sigma_{i_2}^* \sigma_{i_1}^* : H^{1,1}(\overline{\mathcal{S}}(\theta)) \curvearrowright$.*

Proof. We prove the lemma by induction on the length n . It is trivial when $n = 1$. Assume that the lemma holds when the length is $n - 1$. If we put $\sigma' = \sigma_{i_2} \cdots \sigma_{i_n}$, then the induction hypothesis implies that $(\sigma')^* = \sigma_{i_n}^* \cdots \sigma_{i_2}^*$. Lemma 7.2 shows that $I(\sigma_{i_1}) \cap \sigma'(\mathcal{E}(\sigma')) = \{p_{i_1}\} \cap \{p_{i_2}\} = \emptyset$, since p_{i_1} and p_{i_2} are distinct. We now apply Lemma 7.3 to $f = \sigma_{i_1}$ and $g = \sigma'$ to obtain $\sigma^* = (\sigma_{i_1} \sigma')^* = (\sigma')^* \sigma_{i_1}^* = \sigma_{i_n}^* \cdots \sigma_{i_2}^* \sigma_{i_1}^*$. Thus the lemma is true for length n . \square

By Lemma 7.4 the calculation of the action $\sigma^* : H^{1,1}(\overline{\mathcal{S}}(\theta)) \curvearrowright$ is reduced to that of the actions $\sigma_1^*, \sigma_2^*, \sigma_3^* : H^{1,1}(\overline{\mathcal{S}}(\theta)) \curvearrowright$, which is now set forth. Since the cubic surface $\overline{\mathcal{S}}(\theta)$ is rational, we have $H^{1,1}(\overline{\mathcal{S}}(\theta)) = H^2(\overline{\mathcal{S}}(\theta), \mathbb{C})$, where the latter group is described in (32).

Lemma 7.5 *The linear operators $\sigma_1^*, \sigma_2^*, \sigma_3^* : H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \curvearrowright$ have matrix representations as in Table 4 with respect to the basis in (32).*

Proof. First we shall find the matrix representation of σ_1^* . If ξ_{ab} denotes the (a, b) -th entry of the matrix to be found, where $0 \leq a, b \leq 6$, then (33) implies that

$$\sigma_1^* E_b = \sum_{a=0}^6 \xi_{ab} E_a = \sum_{a=0}^6 \delta_a(\sigma_1^* E_b, E_a) E_a,$$

where we put $\delta_a = 1$ for $a = 0$ and $\delta_a = -1$ for $a \neq 0$. Now we claim that

$$\xi_{ab} = \delta_a(\sigma_1^* E_b, E_a), \quad \xi_{ab} = \delta_a \delta_b \xi_{ba}. \quad (49)$$

The first formula in (49) is obvious and the second formula is derived as follows:

$$\xi_{ab} = \delta_a(\sigma_1^* E_b, E_a) = \delta_a(E_b, \sigma_{1*} E_a) = \delta_a(E_b, \sigma_1^* E_a) = (\delta_a \delta_b) \cdot \delta_b(\sigma_1^* E_a, E_b) = (\delta_a \delta_b) \xi_{ba},$$

where in the third equality we have used the fact that σ_1 is an involution; $\sigma_{1*} = (\sigma_1^{-1})^* = \sigma_1^*$. By assertions (1) and (2) of Lemma 6.2 we have $(\sigma_1^* E_1, E_1) = 2$ and $(\sigma_1^* E_1, E_4) = 1$ and likewise $(\sigma_1^* E_4, E_4) = 2$ and $(\sigma_1^* E_4, E_1) = 1$. Then the first formula of (49) yields

$$\xi_{11} = \xi_{44} = -2, \quad \xi_{14} = \xi_{41} = -1. \quad (50)$$

The assertion (3) of Lemma 6.2 together with the second formula of (34) yields

$$\begin{cases} \sigma_1^* E_2 &= 2E_0 - E_1 - E_2 - E_3 - E_4 && - E_6, \\ \sigma_1^* E_3 &= 2E_0 - E_1 - E_2 - E_3 - E_4 - E_5, \\ \sigma_1^* E_5 &= 2E_0 - E_1 && - E_3 - E_4 - E_5 - E_6, \\ \sigma_1^* E_6 &= 2E_0 - E_1 - E_2 && - E_4 - E_5 - E_6, \end{cases} \quad (51)$$

It follows from (50) and (51) that the matrix representation for σ_1^* takes the form

$$\sigma_1^* = \begin{pmatrix} * & * & 2 & 2 & * & 2 & 2 \\ * & -2 & -1 & -1 & -1 & -1 & -1 \\ \bullet & \bullet & -1 & -1 & \bullet & 0 & -1 \\ \bullet & \bullet & -1 & -1 & \bullet & -1 & 0 \\ * & -1 & -1 & -1 & -2 & -1 & -1 \\ \bullet & \bullet & 0 & -1 & \bullet & -1 & -1 \\ \bullet & \bullet & -1 & 0 & \bullet & -1 & -1 \end{pmatrix}, \quad (52)$$

where the entries denoted by \bullet and $*$ are yet to be determined. The entries denoted by \bullet are easily determined by the second formula in (49). The final ingredient taken into account is the fact that σ_1 blows down $L_1 = E_0 - E_1 - E_4$ to the point p_1 (see Lemma 6.1), which leads to

$$\sigma_1^* E_0 - \sigma_1^* E_1 - \sigma_1^* E_4 = 0.$$

This means that the 0-th column is the sum of the first and fourth columns in the matrix (52). Using the second formula in (49) repeatedly, we see that (52) becomes the first matrix of Table 4. The matrix representations of σ_2^* and σ_3^* are obtained just in the same manner. \square

In order to make Lemma 7.5 more transparent, we consider the direct sum decomposition

$$H^2(\overline{\mathcal{S}}(\theta), \mathbb{C}) = V \oplus V^\perp, \quad (53)$$

where V is the subspace spanned by the lines L_1, L_2, L_3 at infinity and V^\perp is the orthogonal complement to it with respect to the intersection form. In view of (31) and (34), we have

$$L_1 = E_0 - E_1 - E_4, \quad L_2 = E_0 - E_2 - E_5, \quad L_3 = E_0 - E_3 - E_6.$$

On the other hand, it is easily seen that the subspace V^\perp is spanned by the vectors

$$2E_0 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6, \quad E_1 - E_4, \quad E_2 - E_5, \quad E_3 - E_6.$$

A little calculation in terms of the new basis shows that Lemma 7.5 can be restated as follows.

Lemma 7.6 *The linear operators $\sigma_1^*, \sigma_2^*, \sigma_3^* : H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \hookrightarrow H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z})$ preserve the subspaces V and V^\perp . They act on these subspaces in the following manner.*

- (1) *The operators $\sigma_1^*, \sigma_2^*, \sigma_3^*$ restricted to V are represented by the matrices*

$$s_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad (54)$$

respectively, with respect to the basis L_1, L_2, L_3 .

- (2) *The operators $\sigma_1^*, \sigma_2^*, \sigma_3^*$ act on V^\perp as the negative of identity -1 .*

It should be noted that each matrix in (54) has eigenvalues 0, 1, 1, counted with multiplicities, and in particular has vanishing determinant.

Theorem 7.7 *The group $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is a universal Coxeter group of rank three over the basic involutions $\sigma_1, \sigma_2, \sigma_3$, that is, there are no relations other than $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$. In particular the expression (46) is unique for any given element $\sigma \in G$.*

Proof. Assume the contrary that there exists a nontrivial relation $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} = 1$ in G such that each neighboring indices i_ν and $i_{\nu+1}$ are distinct. Then it follows from Lemma 7.4 that $\sigma_{i_n}^* \cdots \sigma_{i_2}^* \sigma_{i_1}^* = 1^* = 1$ as a linear endomorphism on $H^2(\overline{\mathcal{S}}(\theta), \mathbb{C})$. But this is impossible because each factor $\sigma_{i_\nu}^*$ has vanishing determinant. This contradiction establishes the theorem. \square

Remark 7.8 Recall that we have introduced the universal Coxeter group G of rank three abstractly in §2. Theorem 7.7 yields a concrete realization of it as a group of birational transformations on the cubic surface $\overline{\mathcal{S}}(\theta)$. Hereafter the former group will be identified with the latter. In this context the 3-dimensional abstract linear space V for the geometric representation $\text{GR} : G \rightarrow \text{O}_B(V)$ in §2 is realized as the subspace of $H^2(\overline{\mathcal{S}}(\theta), \mathbb{C})$ spanned by the lines at infinity L_1, L_2, L_3 . Here we should put $e_1 = L_1, e_2 = L_2, e_3 = L_3$ in accordance with the notation in §2. The symmetric bilinear form B in (11) is now given by the negative of the intersection form on $H^2(\overline{\mathcal{S}}(\theta), \mathbb{C})$ restricted to the subspace V . The basic reflections in (12) are then represented by the matrices

$$r_1 = \begin{pmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}.$$

It is easy to see that the linear operators s_1, s_2, s_3 in (13) have matrix representations as in (54) and hence correspond to the operators $\sigma_1^*, \sigma_2^*, \sigma_3^*$ restricted to V . So the trace $\alpha(\gamma)$ in (14) can be calculated practically by using the matrix representations (54).

Next we shall calculate the characteristic polynomial of the linear map $\sigma^* : H^2(\overline{\mathcal{S}}(\theta), \mathbb{C}) \odot$. In general the characteristic polynomial of a linear endomorphism A is denoted by

$$P(\lambda; A) = \det(\lambda I - A).$$

For the reduced expression (46) of the element σ , we put $s_\sigma := s_{i_n} \cdots s_{i_2} s_{i_1}$ and define

$$\alpha(\sigma) := \text{Tr}[s_\sigma : V \rightarrow V]. \quad (55)$$

Lemma 7.9 *The map σ^* preserves the direct sum decomposition (53) and hence factors as $\sigma^* = (\sigma^*|_V) \oplus (\sigma^*|_{V^\perp})$. The characteristic polynomial of the first component $\sigma^*|_V$ is given by*

$$P(\lambda; \sigma^*|_V) = \begin{cases} \lambda\{\lambda^2 - \alpha(\sigma)\lambda + (-1)^{n-1}\} & (\text{if } i_1 = i_n), \\ \lambda\{\lambda^2 - \alpha(\sigma)\lambda + (-1)^n\} & (\text{if } i_1 \neq i_n). \end{cases} \quad (56)$$

The second component $\sigma^|_{V^\perp}$ is just a scalar operator $(-1)^n$ having the characteristic polynomial*

$$P(\lambda; \sigma^*|_{V^\perp}) = \{\lambda - (-1)^n\}^4.$$

Proof. By Lemma 7.4 we have $\sigma^* = \sigma_{i_n}^* \cdots \sigma_{i_2}^* \sigma_{i_1}^*$. Hence the map σ^* preserves the decomposition (53), because each factor $\sigma_{i_\nu}^*$ does so by Lemma 7.6. Thus there are factorizations $\sigma^* = (\sigma^*|_V) \oplus (\sigma^*|_{V^\perp})$ and $P(\lambda; \sigma^*) = P(\lambda; \sigma^*|_V)P(\lambda; \sigma^*|_{V^\perp})$. The second component $\sigma^*|_{V^\perp}$ is found

$$\sigma^*|_{V^\perp} = (\sigma_{i_n}^*|_{V^\perp}) \cdots (\sigma_{i_1}^*|_{V^\perp}) = (-1)^n.$$

since each factor $\sigma_{i_\nu}^*$ restricted to V^\perp is the scalar operator -1 by assertion (2) of Lemma 7.6.

It remains to consider the first component $\sigma^*|_V$, which is represented by the three-by-three matrix $s_\sigma = s_{i_n} \cdots s_{i_2} s_{i_1}$. The argument will be based on the general fact that the characteristic polynomial of a three-by-three matrix A is given by

$$P(\lambda; A) = \lambda^3 - (\text{Tr } A)\lambda^2 + (\text{Tr } \tilde{A})\lambda - \det A, \quad (57)$$

where \tilde{A} is the adjugate matrix of A , namely, the matrix \tilde{A} such that $A\tilde{A} = \tilde{A}A = (\det A)I$. Let us apply this formula to $A = s_\sigma$. First we have $\text{Tr}(s_\sigma) = \alpha(\sigma)$ by definition (55). Secondly we have $\det(s_\sigma) = 0$, since each factor σ_{i_ν} has vanishing determinant. Finally we wish to calculate the trace $\text{Tr}(\tilde{s}_\sigma)$. The general formula $\widetilde{AB} = \tilde{B}\tilde{A}$ for the product of adjugate matrices yields $\tilde{s}_\sigma = \tilde{s}_{i_1} \tilde{s}_{i_2} \cdots \tilde{s}_{i_n}$. Now it follows from (54) that

$$\tilde{s}_1 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{s}_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{s}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}. \quad (58)$$

Note that among the three rows of the matrix \tilde{s}_i , only the i -th row does not vanish. Thus the only row of \tilde{s}_σ that can be nonzero is the i_1 -th row, so that the trace $\text{Tr}(\tilde{s}_\sigma)$ is just given by the (i_1, i_1) -th entry of \tilde{s}_σ . Now the latter quantity is calculated as

$$(\tilde{s}_{i_1})_{i_1 i_2} (\tilde{s}_{i_2})_{i_2 i_3} \cdots (\tilde{s}_{i_{n-1}})_{i_{n-1} i_n} (\tilde{s}_{i_n})_{i_n i_1},$$

where $(\tilde{s}_i)_{ij}$ denotes the (i, j) -th entry of the matrix \tilde{s}_i . It follows from (58) that $(\tilde{s}_i)_{ij}$ is $+1$ or -1 according as the indices i and j are equal or not. Since i_ν and $i_{\nu+1}$ are distinct for every $\nu \in \{1, \dots, n-1\}$, we have $\text{Tr}(\tilde{s}_\sigma) = (-1)^{n-1}$ or $\text{Tr}(\tilde{s}_\sigma) = (-1)^n$ according as i_n and i_1 are equal or not. Putting all these considerations into (57) yields formula (56). \square

8 Ergodic Properties

We continue to study the dynamical properties of each individual transformation $\sigma \in G$. The main concern in this section is the investigation into the ergodic properties of this map, where the notions of dynamical degree, entropy and invariant measure play important roles. It is a good application of the fundamental methods and techniques in bimeromorphic (or birational) surface dynamics, recently developed by [2, 7, 8, 10]. Since they are not so familiar in the circle of Painlevé equations, we shall develop our discussion upon reviewing some rudiments of them.

We begin with the concept of first dynamical degree [7]. Given a bimeromorphic map f of a compact Kähler surface S , its *first dynamical degree* $\lambda_1(f)$ is defined by

$$\lambda_1(f) := \lim_{N \rightarrow \infty} \|(f^N)^*\|^{1/N},$$

where $\|\cdot\|$ is an operator norm on $\text{End } H^{1,1}(S)$. It is known that the limit certainly exists, independent of the norm $\|\cdot\|$ chosen, $\lambda_1(f) \geq 1$, and $\lambda_1(f)$ is invariant under bimeromorphic conjugation. It is usually difficult to evaluate this quantity in a simple mean. However there is a distinguished class of maps whose first dynamical degree can be equated to a more tractable quantity. A bimeromorphic map $f : S \dashrightarrow S$ is said to be *analytically stable* (AS for short) if the condition $(f^n)^* = (f^*)^n : H^{1,1}(S) \dashrightarrow H^{1,1}(S)$ holds for every $n \in \mathbb{N}$. Evidently, if f is AS then

$$\lambda_1(f) = \text{SR}(f^*), \tag{59}$$

where $\text{SR}(f^*)$ is the spectral radius of the linear endomorphism $f^* : H^{1,1}(S) \dashrightarrow H^{1,1}(S)$. It is known that any bimeromorphic map is bimeromorphically conjugate to an AS map. It is also known that a bimeromorphic map f is AS if and only if

$$\bigcup_{N \geq 0} f^{-N} I(f) \cap \bigcup_{N \geq 0} f^N I(f^{-1}) = \emptyset. \tag{60}$$

This condition may be viewed as a separation between the obstructions to forward and backward dynamics. Back to our context, it is natural to ask when a given element $\sigma \in G$ is AS.

Lemma 8.1 *An element $\sigma \in G$ is AS if and only if the initial index i_1 and the terminal index i_n are distinct in the reduced expression (46) of σ .*

Proof. If σ is AS then it follows from condition (60) that $I(\sigma) \cap I(\sigma^{-1}) = \emptyset$. On the other hand, Lemma 7.2 implies that $I(\sigma) = \{p_{i_n}\}$ and $I(\sigma^{-1}) = \{p_{i_1}\}$. Hence the points p_{i_1} and p_{i_n} must be distinct, that is, the indices i_1 and i_n must be distinct. Conversely, assuming that the indices i_1 and i_n are distinct, we shall show that for every $N \geq 0$,

$$\sigma^{-N}I(\sigma) = \{p_{i_n}\}, \quad \sigma^N I(\sigma^{-1}) = \{p_{i_1}\}. \quad (61)$$

It suffices to verify the first formula of (61), since the second formula is obtained from the first one by replacing σ with σ^{-1} . Since $I(\sigma) = \{p_{i_n}\}$ by Lemma 7.2, we have only to show that $\sigma^{-1}(p_{i_n}) = p_{i_n}$, namely, that the indeterminacy point p_{i_n} of σ is a fixed point of $\sigma^{-1} = \sigma_{i_n} \cdots \sigma_{i_2} \sigma_{i_1}$. By Lemma 6.1, if two indices i and j are distinct, then the point p_i lies on the line L_j and hence is sent to p_j by the map σ_j . Using this fact repeatedly, we see that

$$p_{i_n} \xrightarrow{\sigma_{i_1}} p_{i_1} \xrightarrow{\sigma_{i_2}} p_{i_2} \longrightarrow \cdots \longrightarrow p_{i_{n-1}} \xrightarrow{\sigma_{i_n}} p_{i_n},$$

because every neighboring indices are distinct. Now it follows from formula (61) that σ satisfies condition (60) and hence is AS as desired. \square

Definition 8.2 We introduce two simple examples of AS transformations in G .

- (1) An AS element $\sigma \in G$ is said to be *elementary* if $\sigma = (\sigma_i \sigma_j)^m$ for some $\{i, j, k\} = \{1, 2, 3\}$ and $m \in \mathbb{N}$; otherwise, σ is said to be *non-elementary*.
- (2) An element $\sigma \in G$ is called a *Coxeter element* if $\sigma = \sigma_i \sigma_j \sigma_k$ for some $\{i, j, k\} = \{1, 2, 3\}$.

We may assume without loss of generality that σ is AS, since if σ is not AS then it can be replaced with its conjugate $\sigma' := \tau^{-1} \sigma \tau = \sigma_{i_{\nu+1}} \cdots \sigma_{i_{n-\nu}}$ which is AS, where $\tau = \sigma_{i_1} \cdots \sigma_{i_\nu}$ with ν being the index such that $i_1 = i_n, i_2 = i_{n-1}, \dots, i_\nu = i_{n-\nu+1}$ but $i_{\nu+1} \neq i_{n-\nu}$. Under this assumption we can apply formula (59) to conclude that the first dynamical degree of σ is equal to the spectral radius of the linear map $\sigma^* : H^2(\overline{\mathcal{S}}(\theta), \mathbb{C}) \rightarrow H^2(\overline{\mathcal{S}}(\theta), \mathbb{C})$. On the other hand, Lemma 7.9 implies that the eigenvalues of σ^* are 0, $(-1)^n$ and the roots of the quadratic equation

$$\lambda^2 - \alpha(\sigma)\lambda + (-1)^n = 0, \quad (62)$$

so that the spectral radius of σ^* is the largest absolute value of the roots of equation (62). This observation leads us to investigate the value distribution of $\alpha(\sigma)$.

Lemma 8.3 Assume that $\sigma \in G$ is AS. Then $\alpha(\sigma)$ is an even positive integer. Moreover,

- (1) $\alpha(\sigma) = 2$ if and only if σ is elementary in the sense of Definition 8.2,
- (2) $\alpha(\sigma) = 4$ if and only if σ is a Coxeter element,
- (3) $\alpha(\sigma) = 6$ if and only if $\sigma = \sigma_i \sigma_j \sigma_k \sigma_j$ or $\sigma = \sigma_j \sigma_i \sigma_j \sigma_k$ for some $\{i, j, k\} = \{1, 2, 3\}$.

Proof. Let $\sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n}$ be the reduced expression of σ as in (46). For $\nu = 1, \dots, n$, we put $A_\nu := s_{i_\nu} \cdots s_{i_2} s_{i_1}$ and denote its (i, j) -th entry by $(A_\nu)_{ij}$. By definition (55) we have $\alpha(\sigma) = \text{Tr } A_n$. We may assume that $i_1 = 1$, since the other cases can be treated in a similar manner. In this case, if we put $M_\nu := \min\{(A_\nu)_{ij} : i = 1, 2, 3, j = 2, 3\}$, then

$$M_{\nu+1} \geq M_\nu \quad (\nu = 1, \dots, n-1). \quad (63)$$

Moreover, if the index $j_{\nu+1}$ is defined by $\{j_{\nu+1}\} = \{1, 2, 3\} - \{i_\nu, i_{\nu+1}\}$, then

$$\text{Tr } A_{\nu+1} = \text{Tr } A_\nu + 2(A_\nu)_{j_{\nu+1}, i_{\nu+1}} \quad (\nu = 1, \dots, n-1). \quad (64)$$

Indeed it is easy to see from formula (54) that when $i_1 = 1$, the matrix A_ν takes the form

$$A_\nu = \begin{pmatrix} 0 & a_{12}^\nu & a_{13}^\nu \\ 0 & a_{22}^\nu & a_{23}^\nu \\ 0 & a_{32}^\nu & a_{33}^\nu \end{pmatrix},$$

where a_{ij}^ν , $i = 1, 2, 3$, $j = 2, 3$, are nonnegative integers, and $A_{\nu+1} = s_{i_{\nu+1}} A_\nu$ is given by

$$\begin{aligned} A_{\nu+1} &= \begin{pmatrix} 0 & a_{22}^\nu + a_{32}^\nu & a_{23}^\nu + a_{33}^\nu \\ 0 & a_{22}^\nu & a_{23}^\nu \\ 0 & a_{32}^\nu & a_{33}^\nu \end{pmatrix} & (\text{if } i_{\nu+1} = 1), \\ A_{\nu+1} &= \begin{pmatrix} 0 & a_{12}^\nu & a_{13}^\nu \\ 0 & a_{12}^\nu + a_{32}^\nu & a_{13}^\nu + a_{33}^\nu \\ 0 & a_{32}^\nu & a_{33}^\nu \end{pmatrix} & (\text{if } i_{\nu+1} = 2), \\ A_{\nu+1} &= \begin{pmatrix} 0 & a_{12}^\nu & a_{13}^\nu \\ 0 & a_{22}^\nu & a_{23}^\nu \\ 0 & a_{12}^\nu + a_{22}^\nu & a_{13}^\nu + a_{23}^\nu \end{pmatrix} & (\text{if } i_{\nu+1} = 3). \end{aligned}$$

Inequality (63) readily follows from these observations and formula (64) is verified by a case-by-case check. Indeed, if $i_\nu = 1$ and $i_{\nu+1} = 2$, then $j_{\nu+1} = 3$ and $a_{12}^\nu = a_{22}^\nu + a_{32}^\nu$ so that

$$\text{Tr } A_{\nu+1} = a_{12}^\nu + a_{32}^\nu + a_{33}^\nu = (a_{22}^\nu + a_{32}^\nu) + a_{32}^\nu + a_{33}^\nu = a_{22}^\nu + a_{33}^\nu + 2a_{32}^\nu = \text{Tr } A_\nu + 2(A_\nu)_{j_{\nu+1}, i_{\nu+1}}.$$

If $i_\nu = 2$ and $i_{\nu+1} = 1$, then $j_{\nu+1} = 3$ and $(A_\nu)_{j_{\nu+1}, i_{\nu+1}} = 0$ so that

$$\text{Tr } A_{\nu+1} = a_{22}^\nu + a_{33}^\nu = \text{Tr } A_\nu = \text{Tr } A_\nu + 2(A_\nu)_{j_{\nu+1}, i_{\nu+1}}.$$

The remaining cases can be treated in similar manners. Note that (64) yields an inequality $\text{Tr } A_{\nu+1} \geq \text{Tr } A_\nu$, since $(A_\nu)_{j_{\nu+1}, i_{\nu+1}}$ is nonnegative. A repeated use of formula (64) shows that $\alpha(\sigma) = \text{Tr } A_n$ is an even integer not smaller than 2, because $\text{Tr } A_1 = \text{Tr } s_1 = 2$.

Next we observe that $\alpha(\sigma) = 2$ if $\sigma = (\sigma_i \sigma_j)^m$ for any $\{i, j, k\} = \{1, 2, 3\}$ and $m \in \mathbb{N}$. Indeed, since we are assuming that $i_1 = 1$, we have only to check the two cases where $A_n = (s_2 s_1)^m$ and $A_n = (s_3 s_1)^m$ with $m \in \mathbb{N}$. In either case we have $\alpha(\sigma) = \text{Tr } A_n = 2$ because

$$(s_2 s_1)^m = \begin{pmatrix} 0 & 1 & 2m-1 \\ 0 & 1 & 2m \\ 0 & 0 & 1 \end{pmatrix}, \quad (s_3 s_1)^m = \begin{pmatrix} 0 & 2m-1 & 1 \\ 0 & 1 & 0 \\ 0 & 2m & 1 \end{pmatrix}.$$

From now on we assume that σ is not of the form $(\sigma_i \sigma_j)^m$ for any $\{i, j, k\} = \{1, 2, 3\}$ and $m \in \mathbb{N}$. Then the length n must be not less than 3 and there exists an index ν such that $\{i_\nu, i_{\nu+1}, i_{\nu+2}\} = \{1, 2, 3\}$. Here we may assume without loss of generality that $\nu = 1$, since the quantity $\alpha(\sigma)$ is invariant under any cyclic permutation of the indices (i_1, \dots, i_n) , provided that σ is an AS element. Since moreover we are assuming that $i_1 = 1$, we have

$$A_3 = s_3 s_2 s_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}, \quad \text{or} \quad A_3 = s_2 s_3 s_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix}. \quad (65)$$

If $n = 3$ we have $\sigma = \sigma_1\sigma_2\sigma_3$ or $\sigma = \sigma_1\sigma_3\sigma_2$. Then formula (65) yields $\alpha(\sigma) = \text{Tr } A_3 = 4$ in either case. If $n = 4$ we have $\sigma = \sigma_1\sigma_2\sigma_3\sigma_2$ or $\sigma = \sigma_1\sigma_3\sigma_2\sigma_3$. Since

$$A_4 = s_2s_3s_2s_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 2 & 3 \end{pmatrix}, \quad \text{or} \quad A_4 = s_3s_2s_3s_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix}, \quad (66)$$

we have $\alpha(\sigma) = \text{Tr } A_4 = 6$ in either case. Finally we assume that $n \geq 5$. Then A_4 is either (66) or $A_4 = s_1s_js_ks_1$ for some $\{j, k\} = \{2, 3\}$. In the latter case we must have $A_5 = s_js_1s_js_ks_1$ or $A_5 = s_ks_1s_js_ks_1$. Here we can eliminate the last term s_1 by taking a cyclic permutation of the indices (i_1, \dots, i_n) and obtain $A_4 = s_js_1s_js_k$ or $A_4 = s_ks_1s_js_k$. By relabeling the indices, the matrix A_4 can be reduced to the form (66). So we have only to consider the former case (66). Since σ is assumed to be AS, the index i_n is different from $i_1 = 1$ so that $(A_{n-1})_{j_n, i_n} \geq M_{n-1}$ by the definition of M_ν . Then it follows from (63) and (64) that $\alpha(\sigma) = \text{Tr } A_n$ is estimated as

$$\alpha(\sigma) = \text{Tr } A_{n-1} + 2(A_{n-1})_{j_n, i_n} \geq \text{Tr } A_{n-1} + 2M_{n-1} \geq \text{Tr } A_4 + 2M_4 = 6 + 2 \times 1 = 8.$$

Putting all these arguments together we establish the lemma. \square

Lemma 8.4 *If $\sigma \in G$ is AS then the first dynamical degree of σ is given by*

$$\lambda_1(\sigma) = \frac{1}{2} \left\{ \alpha(\sigma) + \sqrt{\alpha(\sigma)^2 + 4(-1)^{n+1}} \right\}, \quad (67)$$

where $n = \ell_G(\sigma)$ is the length of the element σ . Moreover,

- (1) if σ is elementary then $\lambda_1(\sigma) = 1$,
- (2) if σ is a Coxeter element then $\lambda_1(\sigma) = 2 + \sqrt{5}$,
- (3) if $\sigma = \sigma_i\sigma_j\sigma_k\sigma_j$ or $\sigma = \sigma_j\sigma_i\sigma_j\sigma_k$ for some $\{i, j, k\} = \{1, 2, 3\}$, then $\lambda_1(\sigma) = 3 + 2\sqrt{2}$,
- (4) otherwise, we have $\lambda_1(\sigma) \geq 4 + \sqrt{15}$.

Proof. Since $\alpha(\sigma) \geq 2$ by Lemma 8.3, the quadratic equation (62) has the real roots

$$\lambda_{\pm}(\sigma) = \frac{1}{2} \left\{ \alpha(\sigma) \pm \sqrt{\alpha(\sigma)^2 + 4(-1)^{n+1}} \right\}, \quad (68)$$

where $\lambda_+(\sigma) \geq 1$ and $|\lambda_-(\sigma)| = \lambda_+(\sigma)^{-1} \leq 1$. Therefore the root $\lambda_+(\sigma)$ gives the spectral radius $\text{SR}(\sigma^*)$ of σ^* and hence the first dynamical degree $\lambda_1(\sigma)$ of σ by formula (59). Assertions (1), (2), (3) can be checked directly by using Lemma 8.3. Finally we shall show assertion (4). In this case, since $\alpha(\sigma) \geq 8$ by Lemma 8.3, formula (67) implies that

$$\lambda_1(\sigma) \geq \frac{1}{2} \left\{ \alpha(\sigma) + \sqrt{\alpha(\sigma)^2 - 4} \right\} \geq \frac{1}{2} \left(8 + \sqrt{8^2 - 4} \right) = 4 + \sqrt{15}.$$

Hence the lemma is proved. \square

We proceed to the construction of a natural σ -invariant measure for an AS element $\sigma \in G$. Again let us start with the general situation where $f : S \dashrightarrow S$ is an AS bimeromorphic map on a compact Kähler surface S . If $\lambda_1(f) = 1$ then either f is a dynamically trivial automorphism

or f preserves a rational or elliptic fibration and exhibits an essentially 1-dimensional dynamic [7]. In our case where $f = \sigma$ and $S = \overline{\mathcal{S}}(\theta)$, the condition $\lambda_1(\sigma) = 1$ means that σ is elementary by Lemma 8.4. If so, the existence of a σ -invariant rational fibration on $\overline{\mathcal{S}}(\theta)$ can be seen easily (see Remark 10.3). So we are not interested in the case $\lambda_1(f) = 1$ and assume hereafter that

$$\lambda_1(f) > 1. \quad (69)$$

In this case it is known [7] that there are positive closed $(1, 1)$ -currents μ^\pm on S such that

$$(f^{\pm 1})^* \mu^\pm = \lambda_1(f) \mu^\pm,$$

where μ^+ and μ^- are called the stable and unstable currents for f . A natural strategy to obtain an f -invariant measure μ on S is to take the wedge product

$$\mu = \mu^+ \wedge \mu^-. \quad (70)$$

However the main issue here is whether the operation of wedge product is feasible or not. If the stable and unstable currents are expressed as $\mu^\pm = dd^c g^\pm$ in terms of local potentials g^\pm , then the wedge product (70) may be interpreted as the complex Monge-Ampère operator $dd^c g^+ \wedge dd^c g^-$. In order for this operation to be well-defined, a quantitative condition

$$\sum_{N=0}^{\infty} \lambda_1(f)^{-N} \log \text{dist}(f^N I(f^{-1}), I(f)) > -\infty, \quad (71)$$

is introduced in [2], where dist is the distance on S induced from a Riemannian metric on it. This condition is slightly stronger than (60) and a map enjoying this condition might be called quantitatively AS. Under these settings the following theorem is established in [2].

Theorem 8.5 *If $f : S \curvearrowright$ satisfies conditions (69) and (71), then the wedge product μ of the stable and unstable currents μ^\pm in (70) is well defined and, after a suitable renormalization, μ gives an f -invariant Borel probability measure such that all the conditions in Definition 1.1 are satisfied. Moreover the measure μ puts no mass on any algebraic curve on S .*

Applying this theorem to our situation, we obtain the following theorem.

Theorem 8.6 *For any non-elementary AS map $\sigma \in G$ there exists the wedge product $\mu_\sigma = \mu_\sigma^+ \wedge \mu_\sigma^-$ of the stable and unstable currents μ_σ^\pm for σ and, after a suitable renormalization, μ_σ gives a σ -invariant Borel probability measure such that all the conditions in Definition 1.1 are satisfied. Moreover the measure μ_σ puts no mass on any algebraic curve on $\overline{\mathcal{S}}(\theta)$.*

Proof. It is enough to check that any non-elementary AS map $\sigma \in G$ satisfies conditions (69) and (71). Lemma 8.4 implies that $\lambda_1(\sigma) > 1$ if and only if σ is non-elementary, so that condition (69) is satisfied. In order to check condition (71) let $\sigma = \sigma_{i_1} \cdots \sigma_{i_n}$ be the reduced expression of σ . Since σ is assumed to be AS, the indices i_1 and i_n are distinct and hence $\text{dist}(p_{i_1}, p_{i_n}) > 0$. On the other hand, by formula (61), we have $I(\sigma) = \{p_{i_n}\}$ and $\sigma^N I(\sigma^{-1}) = \{p_{i_1}\}$ independently of $N \geq 0$. Therefore we have

$$\begin{aligned} \sum_{N=0}^{\infty} \lambda_1(\sigma)^{-N} \log \text{dist}(\sigma^N I(\sigma^{-1}), I(\sigma)) &= \log \text{dist}(p_{i_1}, p_{i_n}) \sum_{N=0}^{\infty} \lambda_1(\sigma)^{-N} \\ &= \frac{\lambda_1(\sigma) \log \text{dist}(p_{i_1}, p_{i_n})}{\lambda_1(\sigma) - 1} > -\infty, \end{aligned}$$

which shows that condition (71) is satisfied. The theorem then follows from Theorem 8.5. \square

Remark 8.7 Under the setting of Theorem 8.5 it is shown in [2] that the Lyapunov exponents $L_{\pm}(f)$ of f with respect to the ergodic measure μ satisfy the estimate

$$L_{-}(f) \leq -\frac{\log \lambda_1(f)}{8} < 0 < \frac{\log \lambda_1(f)}{8} \leq L_{+}(f),$$

which applies to the mapping $\sigma : \mathcal{S}(\theta) \curvearrowright$ in Theorem 8.6. On the other hand, we have $L_{-}(\sigma) = -L_{+}(\sigma)$ since σ is area-preserving with respect to the Poincaré residue $\omega(\theta)$ in (28). It follows from the above estimate that $L_{+}(\sigma) \geq \frac{1}{8} \log \lambda_1(\sigma)$.

Finally we shall calculate the entropy of a non-elementary AS map $\sigma \in G$. For a birational map $f : S \curvearrowright$ of a projective surface S and an f -invariant Borel probability measure μ on S , there are two concepts of entropies: one is the *measure-theoretic entropy* $h_{\mu}(f)$ with respect to the invariant measure μ and the other is the *topological entropy* $h_{\text{top}}(f)$. In general these quantities and the first dynamical degree $\lambda_1(f)$ are related as

$$h_{\mu}(f) \leq h_{\text{top}}(f) \leq \log \lambda_1(f), \quad (72)$$

where the first inequality is the so-called variational principle and the second inequality is a consequence of a main result of [8]. Moreover, if f satisfies conditions (69) and (71) and if μ is the invariant measure mentioned in Theorem 8.5, then it is proved in [10] that the leftmost and rightmost terms in (72) are equal and consequently all the three terms in (72) coincide. Applying this triple coincidence to our situation we obtain the following theorem.

Theorem 8.8 *For any non-elementary AS map $\sigma \in G$, we have*

$$h_{\mu_{\sigma}}(\sigma) = h_{\text{top}}(\sigma) = \log \lambda_1(\sigma). \quad (73)$$

where μ_{σ} is the σ -invariant probability measure mentioned in Theorem 8.6. The value of (73) is not smaller than $\log(2 + \sqrt{5})$ with equality if and only if σ is a Coxeter element.

Proof. The proof is already finished in the above argument. The assertion that (73) takes its minimum precisely when σ is a Coxeter element follows from Lemma 8.4. \square

Remark 8.9 Theorems 8.6 and 8.8 are results for an element $\sigma \in G$ viewed as a birational map of the projective surface $\overline{\mathcal{S}}(\theta)$. However, since the invariant measure μ_{σ} put no mass on any algebraic curve on $\overline{\mathcal{S}}(\theta)$, the lines $L = L_1 \cup L_2 \cup L_3$ at infinity can be neglected as far as the ergodic properties of $\sigma : \overline{\mathcal{S}}(\theta) \curvearrowright$ relative to the measure μ_{σ} are concerned. So Theorems 8.6 and 8.8 lead to results for the biregular map $\sigma' := \sigma|_{\mathcal{S}(\theta)}$ of the affine surface $\mathcal{S}(\theta) = \overline{\mathcal{S}}(\theta) - L$. Namely μ_{σ} can be restricted without losing any mass to an σ' -invariant Borel probability measure $\mu_{\sigma'}$ on $\mathcal{S}(\theta)$ such that the conditions in Definition 1.1 are satisfied, and one has an equality $h_{\mu_{\sigma'}}(\sigma') = \log \lambda_1(\sigma)$. Here we do not refer to $h_{\text{top}}(\sigma')$, because the concept of topological entropy, usually defined on a compact space, is not very clear on the affine surface $\mathcal{S}(\theta)$. In what follows σ' and $\mu_{\sigma'}$ will be written σ and μ_{σ} for the simplicity of notation.

9 Number of Periodic Points

Given any non-elementary AS element $\sigma \in G$, we are interested in the number of periodic points of the birational map $\sigma : \overline{\mathcal{S}}(\theta) \curvearrowright$. For each positive integer $N \in \mathbb{N}$ we shall consider the

set of all periodic points of period N on the projective cubic surface $\overline{\mathcal{S}}(\theta)$,

$$\overline{\text{Per}}_N(\sigma; \theta) := \{ X \in \overline{\mathcal{S}}(\theta) - I(\sigma^N) : \sigma^N(X) = X \},$$

as well as the corresponding set on the affine cubic surface $\mathcal{S}(\theta)$,

$$\text{Per}_N(\sigma; \theta) := \{ x \in \mathcal{S}(\theta) : \sigma^N(x) = x \}.$$

Our tasks are then to count the cardinality of $\overline{\text{Per}}_N(\sigma; \theta)$ and to relate it with the cardinality of $\text{Per}_N(\sigma; \theta)$. The first task is based on the Lefschetz fixed point formula, while the second one is by a careful inspection of the behavior of the map σ around the lines $L = L_1 \cup L_2 \cup L_3$ at infinity. In order to apply the Lefschetz fixed point formula, we need the following lemma.

Lemma 9.1 *Assume that $\sigma \in G$ is AS and non-elementary. Then for any $N \in \mathbb{N}$, the birational map $\sigma : \overline{\mathcal{S}}(\theta) \dashrightarrow \overline{\mathcal{S}}(\theta)$ admits no curves of periodic points of period N .*

Proof. The lemma is proved by contradiction. Assume that σ admits a curve (an effective divisor) $D \subset \overline{\mathcal{S}}(\theta)$ of periodic points of some period N . Since σ^N fixes D pointwise, we have $(\sigma^N)^*D = D$ in $H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z})$. Moreover, since σ is assumed to be AS, we have $(\sigma^*)^N = (\sigma^N)^*$ and hence $(\sigma^*)^N D = D$, which means that $(\sigma^*)^N$ has an eigenvalue 1 with an eigenvector D . On the other hand, by Lemma 7.9, there is a direct sum decomposition $\sigma^* = (\sigma^*|_V) \oplus (\sigma^*|_{V^\perp})$ with $H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) = V \oplus V^\perp$ as in (53) such that $\sigma^*|_V$ has the eigenvalues 0 and $\lambda_\pm(\sigma)$ as in (68), while $\sigma^*|_{V^\perp}$ is the scalar operator $(-1)^n$ on V^\perp , where $n = \ell_G(\sigma)$ is the length of σ . By Lemma 8.4 we have $\lambda_+(\sigma) = \lambda_1(\sigma) > 1$ and $|\lambda_-(\sigma)| = \lambda_1(\sigma)^{-1} < 1$, since σ is assumed to be non-elementary. Therefore the eigenvector D of σ^* must belong to the subspace V^\perp and its eigenvalue 1 must arise as the N -th power of the scalar operator $(-1)^n = \sigma^*|_{V^\perp}$, where the integer nN must be even. Since $L_1, L_2, L_3 \in V$ and $D \in V^\perp$, we have

$$(D, L_1) = (D, L_2) = (D, L_3) = 0. \quad (74)$$

We now write $D = D' + m_1 L_1 + m_2 L_2 + m_3 L_3$, where D' is either empty or an effective divisor not containing L_1, L_2, L_3 as an irreducible component of it and m_1, m_2, m_3 are nonnegative integers. Since $(L_i, L_j) = -1$ for $i = j$ and $(L_i, L_j) = 1$ for $i \neq j$, formula (74) yields

$$\begin{aligned} 0 &= (D, L_1) = (D', L_1) - m_1 + m_2 + m_3, \\ 0 &= (D, L_2) = (D', L_2) + m_1 - m_2 + m_3, \\ 0 &= (D, L_3) = (D', L_3) + m_1 + m_2 - m_3, \end{aligned}$$

which sum up to

$$(D', L_1) + (D', L_2) + (D', L_3) + m_1 + m_2 + m_3 = 0. \quad (75)$$

Since none of the lines L_1, L_2, L_3 is an irreducible component of D' , the intersection number (D', L_i) must be nonnegative for every $i = 1, 2, 3$. Since the numbers m_1, m_2, m_3 are also nonnegative, formula (75) implies that $(D', L_1) = (D', L_2) = (D', L_3) = 0$ and $m_1 = m_2 = m_3 = 0$. Hence $D = D'$ and $(D, L_1) = (D, L_2) = (D, L_3) = 0$. It follows that D is an effective divisor such that $(D, L_i) = 0$ and L_i is not an irreducible component of D for every $i = 1, 2, 3$. This means that the compact curve D does not intersect $L = L_1 \cup L_2 \cup L_3$ and hence must lie

in the affine cubic surface $\mathcal{S}(\theta) = \overline{\mathcal{S}}(\theta) - L$. However no compact curve can lie in any affine variety. By this contradiction the lemma is established. \square

Now we shall apply the Lefschetz fixed point formula to the iterates of a non-elementary AS element $\sigma \in G$. For each $N \in \mathbb{Z}$ let $\Gamma_N \subset \overline{\mathcal{S}}(\theta) \times \overline{\mathcal{S}}(\theta)$ be the graph of the N -th iterate $\sigma^N : \overline{\mathcal{S}}(\theta) \rightarrow \overline{\mathcal{S}}(\theta)$, and $\Delta \subset \overline{\mathcal{S}}(\theta) \times \overline{\mathcal{S}}(\theta)$ be the diagonal. Note that $\Gamma_N = \Gamma_{-N}^\vee$, where Γ_{-N}^\vee is the reflection of Γ_{-N} around the diagonal Δ . Moreover let $I_N \subset \overline{\mathcal{S}}(\theta)$ denote the indeterminacy set of σ^N . Then the Lefschetz fixed point formula consists of two equations concerning the intersection number (Γ_N, Δ) of the cycles Γ_N and Δ in $\overline{\mathcal{S}}(\theta) \times \overline{\mathcal{S}}(\theta)$,

$$(\Gamma_N, \Delta) = \sum_{q=0}^4 (-1)^q \text{Tr} [(\sigma^N)^* : H^q(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \rightarrow H^q(\overline{\mathcal{S}}(\theta), \mathbb{Z})], \quad (76)$$

$$(\Gamma_N, \Delta) = \# \overline{\text{Per}}_N(\sigma; \theta) + \sum_{p \in I_N} \mu((p, p), \Gamma_N \cap \Delta), \quad (77)$$

where $\mu((p, p), \Gamma_N \cap \Delta)$ denotes the multiplicity of intersection between Γ_N and Δ at (p, p) . Lemma 9.1 assures that all terms involved in (76) and (77) are well defined and finite.

Lemma 9.2 *Let $n = \ell_G(\sigma)$ be the length of σ . Then formula (76) becomes*

$$(\Gamma_N, \Delta) = \lambda_1(\sigma)^N + (-1)^{nN} \lambda_1(\sigma)^{-N} + 4(-1)^{nN} + 2.$$

Proof. We put $T_N^q = \text{Tr} [(\sigma^N)^* : H^q(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \rightarrow H^q(\overline{\mathcal{S}}(\theta), \mathbb{Z})]$. Because $\overline{\mathcal{S}}(\theta)$ is a smooth rational surface,

$$H^q(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (q = 0, 4), \\ 0 & (q = 1, 3). \end{cases}$$

Trivially we have $T_N^0 = 1$ and $T_N^1 = T_N^3 = 0$. Since σ and so σ^N are birational, we have $T_N^4 = 1$. Since the map σ is assumed to be AS, we have $(\sigma^N)^* = (\sigma^*)^N : H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \rightarrow H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z})$. By Lemmas 7.9 and 8.4, σ^* has three simple eigenvalues 0, $\lambda_+(\sigma) = \lambda_1(\sigma)$, $\lambda_-(\sigma) = (-1)^n \lambda_1(\sigma)^{-1}$ and a quadruple eigenvalue $(-1)^n$. Thus we have $T_N^2 = 0^N + \lambda_1(\sigma)^N + (-1)^{nN} \lambda_1(\sigma)^{-N} + 4(-1)^{nN}$. Substituting these data into (76) yields the assertion of the lemma. \square

Lemma 9.3 *Formula (77) becomes*

$$(\Gamma_N, \Delta) = \# \overline{\text{Per}}_N(\sigma; \theta) + 1 = \# \text{Per}_N(\sigma; \theta) + 2.$$

Proof. Let $\sigma = \sigma_{i_1} \cdots \sigma_{i_n}$ be the reduced expression of σ . Since σ is assumed to be AS, for any $N \in \mathbb{N}$ the reduced expression of σ^N is given by $\sigma^N = \overbrace{\sigma_{i_1} \cdots \sigma_{i_n}} \cdots \overbrace{\sigma_{i_1} \cdots \sigma_{i_n}}^{(N\text{-times})}$. Moreover, since σ is assumed to be non-elementary, the indices $\{i_1, \dots, i_n\}$ range the entire index set $\{1, 2, 3\}$. By Lemma 7.2 the exceptional set of σ^N is given by

$$\mathcal{E}(\sigma^N) = \bigcup_{\nu=1}^n L_{i_\nu} = L_1 \cup L_2 \cup L_3 = L,$$

whose σ^N -image is $\sigma^N(L) = \sigma^N(\mathcal{E}(\sigma^N)) = \{p_{i_1}\}$. This means that p_{i_1} is the unique fixed point of the map σ^N on the lines L at infinity. Lemma 7.2 also implies that p_{i_n} is the unique

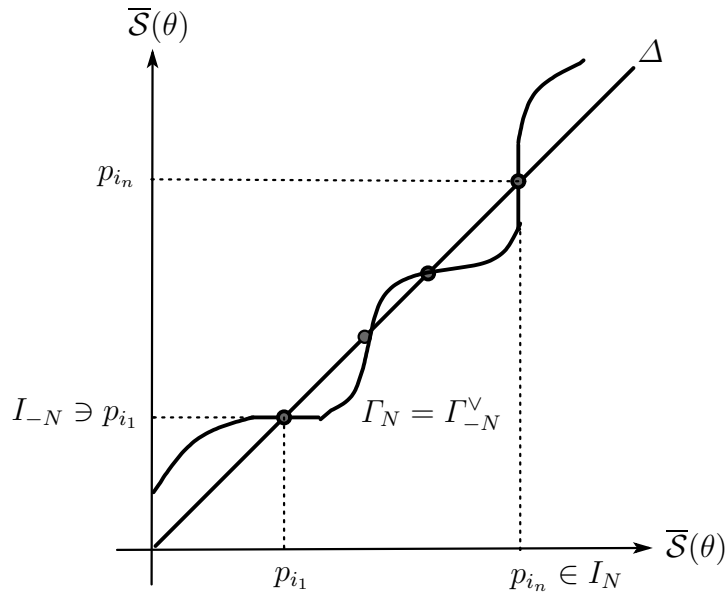


Figure 11: The indeterminacy point p_{i_n} of σ^N is a superattracting fixed point of σ^{-N}

indeterminacy point of σ^N . Therefore we have $\overline{\text{Per}}_N(\sigma; \theta) = \text{Per}_N(\sigma; \theta) \cup \{p_{i_1}\}$ and $I_N = \{p_{i_n}\}$, which implies that formula (77) is rewritten as

$$\begin{aligned} (\Gamma_N, \Delta) &= \# \overline{\text{Per}}_N(\sigma; \theta) + \mu((p_{i_n}, p_{i_n}), \Gamma_N \cap \Delta), \\ \# \overline{\text{Per}}_N(\sigma; \theta) &= \# \text{Per}_N(\sigma; \theta) + \nu(p_{i_1}, \sigma^N), \end{aligned} \quad (78)$$

where $\nu(p_{i_1}, \sigma^N)$ is the local index of the map σ^N around the fixed point p_{i_1} . If j and k are defined by $\{j, k\} = \{1, 2, 3\} - \{i_1\}$, then L_j and L_k are linearly independent lines passing through the point p_{i_1} . These two lines are mapped onto the single point p_{i_1} by σ^N since $\sigma^N(L) = \{p_{i_1}\}$. This implies that p_{i_1} is a superattracting fixed point of σ^N , namely,

$$\nu(p_{i_1}, \sigma^N) = \det(I - (d\sigma^N)_{p_{i_1}}) = \det(I - O) = 1.$$

Likewise p_{i_n} is a superattracting fixed point of $\sigma^{-N} = (\sigma^{-1})^N$ where $\sigma^{-1} = \sigma_{i_n} \cdots \sigma_{i_1}$ is the reduced expression of σ^{-1} (see Figure 11), so that the same reasoning as above with σ replaced by σ^{-1} yields $\nu(p_{i_n}, \sigma^{-N}) = 1$. Therefore we have

$$\mu((p_{i_n}, p_{i_n}), \Gamma_N \cap \Delta) = \mu((p_{i_n}, p_{i_n}), \Gamma_{-N}^V \cap \Delta) = \mu((p_{i_n}, p_{i_n}), \Gamma_{-N} \cap \Delta) = \nu(p_{i_n}, \sigma^{-N}) = 1.$$

These arguments imply that (78) is equivalent to the assertion of the lemma. \square

Putting Lemmas 9.2 and 9.3 together, we have established the following theorem.

Theorem 9.4 *Let $\sigma \in G$ be any non-elementary AS map with length $n = \ell_G(\sigma)$. For any $N \in \mathbb{N}$ the cardinalities of periodic points of period N are finite and explicitly given by*

$$\begin{aligned} \# \overline{\text{Per}}_N(\sigma; \theta) &= \lambda_1(\sigma)^N + (-1)^{nN} \lambda_1(\sigma)^{-N} + 4(-1)^{nN} + 1, \\ \# \text{Per}_N(\sigma; \theta) &= \lambda_1(\sigma)^N + (-1)^{nN} \lambda_1(\sigma)^{-N} + 4(-1)^{nN}. \end{aligned} \quad (79)$$

The numbers grow exponentially as the period N tends to infinity, with the growth rate $\lambda_1(\sigma)$.

10 Back to Painlevé VI

Back to the space of initial conditions for $P_{VI}(\kappa)$ through the Riemann-Hilbert correspondence, we are now able to deduce the dynamical properties of the Poincaré return map for $P_{VI}(\kappa)$ from the already established properties of the dynamical system on the affine cubic surface $\mathcal{S}(\theta)$. This deduction is based on the following lemma.

Lemma 10.1 *Assume that $\kappa \in \mathcal{K} - \mathbf{Wall}$. Given any loop $\gamma \in \pi_1(Z, z)$, let $\sigma \in G(2)$ be the corresponding element via the isomorphism (9). Then the Poincaré return map $\gamma_* : M_z(\kappa) \circlearrowright$ along the loop γ is strictly conjugated to the biregular map $\sigma : \mathcal{S}(\theta) \circlearrowright$ via the Riemann-Hilbert correspondence (26) and the commutative diagram (18).*

Proof. By Theorem 4.1 the Riemann-Hilbert correspondence (26) is biholomorphic under the assumption that $\kappa \in \mathcal{K} - \mathbf{Wall}$. Hence, for $i = 1, 2, 3$, the half-Poincaré map β_{i*} in (21) is strictly conjugate to the transformation g_i in (27). Being squared, β_{i*}^2 is strictly conjugate to g_i^2 . On the other hand, using formulas (27) and (37), one can easily check that

$$g_i^2 = \sigma_i \sigma_{i+1}, \quad (80)$$

where the index should be considered modulo 3. Furthermore, in view of formula (20), the Poincaré return map $\gamma_{i*} : M_z(\kappa) \circlearrowright$ is strictly conjugate to $\beta_{i*}^2 : \mathcal{M}_t(\kappa) \circlearrowright$ via the commutative diagram (18). Then the lemma is established by combining all these observations. \square

The above conjugacy principle stands on the isomorphism of groups $\pi_1(Z, z) \rightarrow G(2)$ in (9), where the abstract group $G(2)$ in §2 is identified with its concrete realization as a group of birational maps on $\overline{\mathcal{S}}(\theta)$ (see Remark 7.8). In order to utilize the results on cubic surface, we need to establish certain relations between the above two groups, e.g., between the minimality of a loop in $\pi_1(Z, z)$ and the analytic stability of an element in $G(2)$, etc.

Lemma 10.2 *Let $\sigma \in G(2)$ be the image of a loop $\gamma \in \pi_1(Z, z)$ under the isomorphism (9).*

- (1) *If γ is minimal in the sense of Definition 2.6, then σ is AS.*
- (2) *If moreover γ is non-elementary in the sense of Definition 1.2, then σ is non-elementary in the sense of Definition 8.2.*

Proof. Let (7) and (8) be the reduced expressions of γ and σ respectively, where $n = 2m$. Assume the contrary that σ is not AS, namely, that $i_1 = i_n$. The argument is separated into two cases: Case 1 where $i_2 = i_{n-1}$ and Case 2 where $i_2 \neq i_{n-1}$. If we define σ' and τ by

$$\sigma' := \begin{cases} \sigma_{i_3} \sigma_{i_4} \cdots \sigma_{i_{n-2}} \\ \sigma_{i_{n-1}} \sigma_{i_2} \cdots \sigma_{i_{n-2}} \end{cases} \quad \tau := \begin{cases} \sigma_{i_1} \sigma_{i_2} & \text{(Case 1),} \\ \sigma_{i_1} \sigma_{i_{n-1}} & \text{(Case 2).} \end{cases}$$

then one has $\sigma = \tau \sigma' \tau^{-1}$ and the length of σ' is given by

$$\ell_G(\sigma') = \begin{cases} n - 4 = 2(m - 2) & \text{(Case 1),} \\ n - 2 = 2(m - 1) & \text{(Case 2).} \end{cases}$$

Let $\gamma', \delta \in \pi_1(Z, z)$ be the loops corresponding to $\sigma', \tau \in G(2)$. Then $\gamma = \delta\gamma'\delta^{-1}$ and

$$\ell_{\pi_1}(\gamma') = \begin{cases} m-2 & \text{(Case 1),} \\ m-1 & \text{(Case 2).} \end{cases}$$

In either case γ' is conjugate to γ and the length of γ' is smaller than that of γ . This contradicts the minimality of γ and hence σ must be AS, which proves assertion (1). Assertion (2) easily follows from Definitions 1.2 and 8.2 and the translation rule (10). \square

We are now in a position to establish our main results, Theorems 2.1, 2.2 and 2.8, together with the related statements in Remarks 2.9 and 2.10. Let $\gamma \in \pi_1(Z, z)$ be any non-elementary loop and $\sigma \in G(2)$ be the corresponding element under the isomorphism (9). As mentioned in Remark 2.5, we may assume without loss of generality that γ is minimal. By Lemma 10.2 the birational map $\sigma : \overline{\mathcal{S}}(\theta) \circlearrowleft$ is AS and non-elementary, so that Theorems 8.6, 8.8 and 9.4 can be applied to the map σ . Then the concluding arguments of this article proceed as follows.

Proof of Theorem 2.1 and (1) of Theorem 2.8. Let μ_σ be the σ -invariant Borel probability measure stated in Theorem 8.6. As is mentioned in Remark 8.9, the measure μ_σ can be restricted to the affine cubic surface $\mathcal{S}(\theta)$ without losing any mass and any ergodic properties. The resulting measure on $\mathcal{S}(\theta)$ is also denoted by μ_σ . We pull it back to the space $M_z(\kappa)$ of initial conditions via the Riemann-Hilbert correspondence. Let μ_γ be the resulting measure on $M_z(\kappa)$. It is now clear from Theorems 8.6 and 8.8 that the measure μ_γ satisfies all the requirements in Theorem 2.1 and in assertion (1) of Theorem 2.8. Here note that formula (67) leads to (15), since the length $n = \ell_G(\sigma)$ of $\sigma \in G(2)$ is an even integer. \square

Proof of Theorem 2.2 and (2) of Theorem 2.8. We have defined in (6) the set $\text{Per}_N(\gamma; \kappa)$ of periodic points of period N for the Poincaré return map γ_* . By Lemma 10.1 the Riemann-Hilbert correspondence (26) maps $\text{Per}_N(\gamma; \kappa)$ bijectively onto $\text{Per}_N(\sigma; \theta)$ and hence

$$\# \text{Per}_N(\gamma; \kappa) = \# \text{Per}_N(\sigma; \theta).$$

Then Theorem 2.2 and assertion (2) of Theorem 2.8 are an immediate consequence of the above equality and Theorem 9.4, where we note that $n = \ell_G(\sigma)$ is even in the formula (79). \square

Remark 10.3 Detailed explanations of Remarks 2.9 and 2.10 are in order at this stage.

- (1) The first half of Remark 2.9 follows from Lemma 8.4. Indeed one has $\lambda_1(\sigma) \geq 3 + 2\sqrt{2}$ for every even non-elementary map $\sigma \in G(2)$. Here one has the equality if and only if $\sigma = \sigma_i\sigma_j\sigma_k\sigma_j$ or $\sigma = \sigma_j\sigma_i\sigma_j\sigma_k$ for some $\{i, j, k\} = \{1, 2, 3\}$. As is easily seen, this occurs precisely when σ comes from an eight-loop in Example 2.3 through the isomorphism (9). For example, if $(i, j, k) = (1, 2, 3)$ then $\sigma = \sigma_1\sigma_2\sigma_3\sigma_2$ comes from the eight-loop $\gamma_1\gamma_2^{-1}$.
- (2) An inspection of formula (27) shows that the transformation $g_i^2 = \sigma_i\sigma_{i+1}$ in (80) preserves the fibration $\mathcal{S}(\theta) \rightarrow \mathbb{C}$, $x = (x_1, x_2, x_3) \mapsto x_k$, where (i, j, k) is the cyclic permutation of $(1, 2, 3)$. Pull it back to the space $M_z(\kappa)$ via the Riemann-Hilbert correspondence. Then the resulting fibration $M_z(\kappa) \rightarrow \mathbb{C}$ is preserved by the Poincaré return map $\gamma_{i*} : M_z(\kappa) \circlearrowleft$ along the i -th basic loop γ_i . This explains the second half of Remark 2.9.
- (3) As is mentioned in Remark 5.1, the Riemann-Hilbert correspondence is an area-preserving biholomorphism between $(M_z(\kappa), \omega(\kappa))$ and $(\mathcal{S}(\theta), \omega(\theta))$ which intertwines the invariant measures μ_γ and μ_σ . Therefore Remark 2.10 readily follows from Remark 8.7.

- (4) In connection with Example 2.3 we give the relation between Pochhammer loops in $\pi_1(Z, z)$ and Coxeter elements in G . For any cyclic permutation (i, j, k) of $(1, 2, 3)$, the Pochhammer loop $\wp = [\gamma_i, \gamma_j^{-1}]$ corresponds to the square c^2 of the Coxeter element $c = \sigma_i \sigma_j \sigma_k$ via the isomorphism (9). Hence one has $\lambda(\wp) = \lambda_1(c^2) = \lambda_1(c)^2 = (2 + \sqrt{5})^2 = 9 + 4\sqrt{5}$, which yields the formula in (2) of Example 2.3.

In this article we have observed that the geometry of cubic surfaces and dynamical systems on them play important parts in understanding an aspect of the global structure of the sixth Painlevé equation. Their relevance to other aspects will be discussed elsewhere (e.g. [21]).

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